Absorbing Boundary Conditions for Free-Surface Flows in Open Domains

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Motivation for Absorbing Boundary Conditions

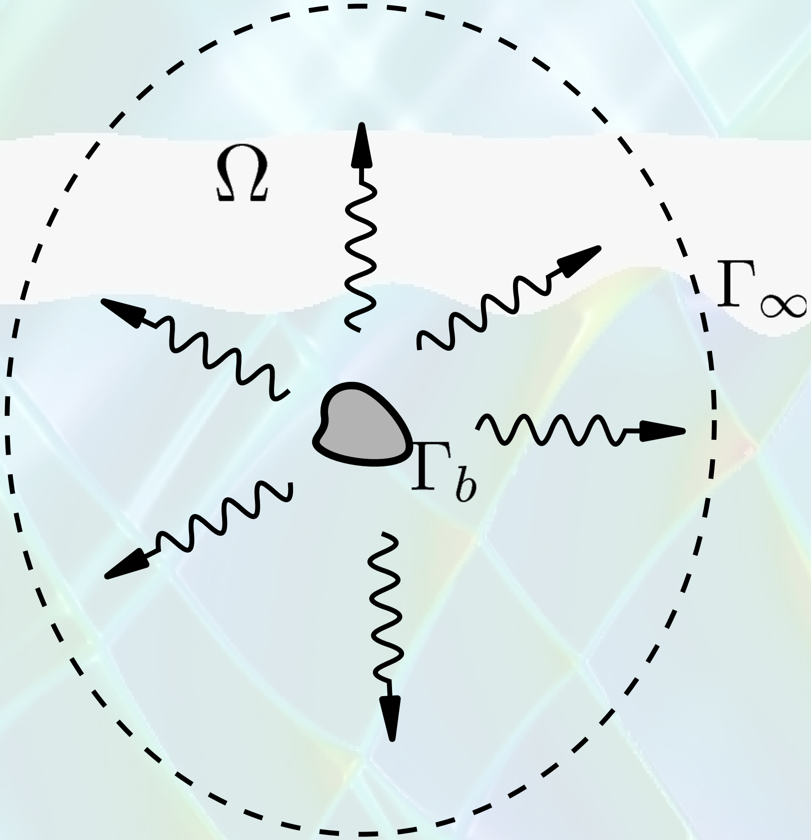
In wave-like propagation problems, not including ABC may lead to **non-convergent solutions**.

$$u_{tt} = c^2 \Delta u, \text{ in } \Omega$$

$$u = \bar{u}(x,t), \text{ at } \Gamma_b, \quad u = 0, \text{ at } \Gamma_\infty v$$

- u doesn't converge to the correct solution irradiating energy from the source, even if $\Gamma_\infty \to \infty$. A *standing wave* is always found.
- u **is unbounded** if $\bar{u}$ emits in an eigenfrequency which is a resonance mode of the closed cavity.

Absorbing boundary conditions must be added to the outer boundary in order to let energy be extracted from the domain.



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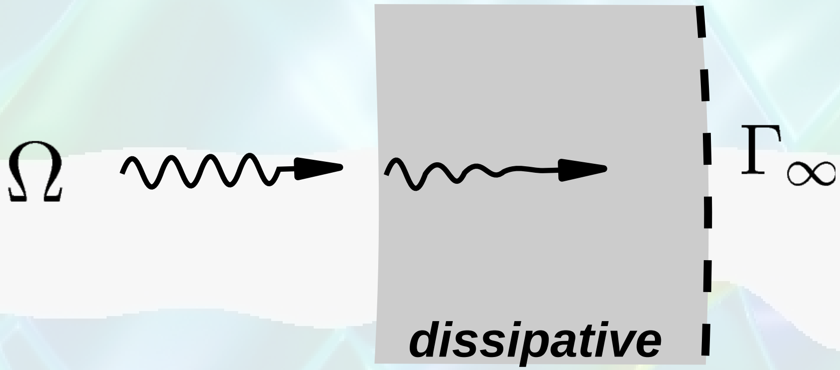
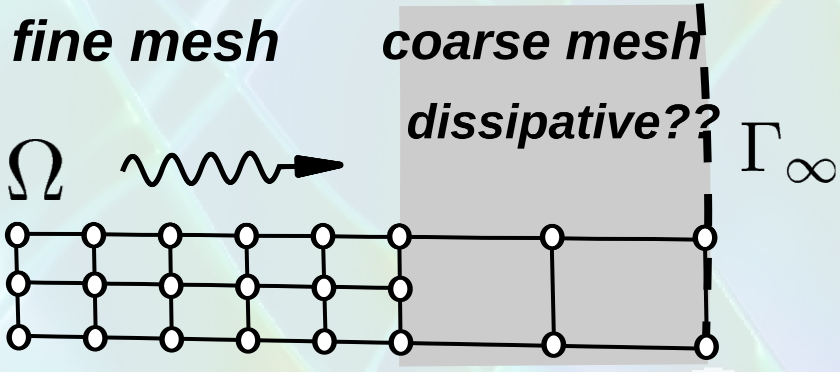
Motivation for Absorbing Boundary Conditions (cont.)

If a dissipative region is large enough so that many wavelengths are included in the region may act as an absorbing layer (AL), i.e. as an absorbing boundary condition (ABC).



However, it is a common misconception to assume that a coarse mesh adds dissipation and consequently may improve absorption. For instance for the wave equation a coarse mesh may lead to evanescent solutions and then to act as a fully reflecting boundary.



Boundary conditions for advective systems

Well known theory and practice for hyperbolic systems say that at a boundary the number of Dirichlet conditions should be equal to the **number of incoming characteristics**.

$$\frac{\partial \mathcal{H}(\mathbf{U})}{\partial t} + \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial x_j} = 0$$

$$A_{c,j} = \frac{\partial \mathcal{F}_{c,j}(\mathbf{U})}{\partial \mathbf{U}}, \quad C = \frac{\partial \mathcal{H}}{\partial \mathbf{U}}$$

Nbr. of incoming characteristics $= \mathrm{sum}(\mathrm{eig}(\mathbf{C}^{-1}\mathbf{A}\cdot\hat{\mathbf{n}}) < 0)$

$\hat{\mathrm{n}}$ is the exterior normal.

Adding extra Dirichlet conditions leads to spurious shocks, and lack of enough Dirichlet conditions leads to instability.

Absorbing boundary conditions

However, this kind of conditions are, generally, **reflective**. Consider a pure advective system of equations in 1D, i.e., $\mathcal{F}_{d,j} \equiv 0$

$$\frac{\partial \mathcal{H}(\mathbf{U})}{\partial t} + \frac{\partial \mathcal{F}_{c,x}(\mathbf{U})}{\partial x} = 0, \text{ in } [0, L]. \tag{1}$$

If the system is *"linear"*, i.e., $\mathcal{F}_{c,x}(\mathbf{U}) = \mathbf{A}\mathbf{U}$, $\mathcal{H}(\mathbf{U}) = \mathbf{C}\mathbf{U}$ ($\mathbf{A}$ and $\mathbf{C}$ do not depend on $\mathbf{U}$), a first order linear system is obtained

$$\mathbf{C}\,\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\,\frac{\partial \mathbf{U}}{\partial x} = 0. \tag{2}$$

If the system is *"hyperbolic"* it is possible to make the following eigenvalue decomposition for $\mathbf{C}^{-1}\mathbf{A}$

$$\mathbf{C}^{-1}\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}, \quad \mathbf{S} \text{ real and invertible, } \mathbf{\Lambda} \text{ real and diagonal.} \tag{3}$$

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Absorbing boundary conditions (cont.)

Introduce *"characteristic variables"* $\mathbf{V} = \mathbf{S}^{-1}\mathbf{U}$, then

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{V}}{\partial x} = 0. \tag{4}$$

Now, each equation is a linear scalar advection equation

$$\frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0, \quad \text{(no summation over } k\text{).} \tag{5}$$

v_k **are the *"characteristic components"* and** λ_k **are the *"characteristic velocities"* of propagation.**

Linear 1D absorbing boundary conditions

Assuming $\lambda_k \neq 0$, the absorbing boundary conditions are, depending on the sign of λ_k,

$$
\begin{aligned}
&\text{if } \lambda_k > 0\text{: } v_k(0) = \bar{v}_{k0}; \quad \text{no boundary condition at } x = L \\
&\text{if } \lambda_k < 0\text{: } v_k(L) = \bar{v}_{kL}; \quad \text{no boundary condition at } x = 0
\end{aligned}
\tag{6}
$$

This can be put in compact form as

$$
\begin{aligned}
\mathbf{\Pi}_V^+(\mathbf{V} - \bar{\mathbf{V}}_0) = 0; \quad &\text{at } x = 0 \\
\mathbf{\Pi}_V^-(\mathbf{V} - \bar{\mathbf{V}}_L) = 0; \quad &\text{at } x = L
\end{aligned}
\tag{7}
$$

- Impose incoming characteristics.
- Let free outgoing characteristics.

Linear 1D absorbing boundary conditions (cont.)

Or, coming back to the boundary condition at $x = L$ in the U basis, it can be written

$$\mathbf{\Pi}_V^- \, \mathbf{S}^{-1}(\mathbf{U} - \bar{\mathbf{U}}_L) = 0 \tag{8}$$

or, multiplying by S at the left

$$\mathbf{\Pi}_U^\pm \, (\mathbf{U} - \bar{\mathbf{U}}_{0,L}) = 0, \quad \textbf{at } x = 0, L, \tag{9}$$

where

$$\mathbf{\Pi}_U^\pm = \mathbf{S} \, \mathbf{\Pi}_V^\pm \, \mathbf{S}^{-1}, \tag{10}$$

Absorbing layer (AL)

If, for the *first order* aborbing boundary condition we condense the Lagrange multiplier version, we get a penalized version of the form

$$\mathbf{C}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + K\mathbf{H}\mathbf{U} = 0, \quad \mathbf{H} = \frac{1}{\epsilon}\mathbf{C}\mathbf{\Pi}_U^+ \tag{11}$$

where H is the *"absorbing layer matrix"*. $K = K(x)$ is a factor controlling the intensity of the AL.

Also a combination of $\mathbf{\Pi}_U^+$ and $\mathbf{\Pi}_U^-$ can be chosen. The simplest choice is $\mathbf{H} = K\mathbf{C}(\mathbf{\Pi}_U^+ + \mathbf{\Pi}_U^-) = \mathbf{C}\mathbf{I} = \mathbf{C}$.

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Absorbing layer for shallow-water

We assume that in the simulation of the FS flow, near the boundary the flow can be approximated by the shallow water equations, which are

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{G}, \tag{12}$$

where

$$\begin{aligned} \mathbf{U} &= [h, u]^T, \\ \mathbf{F}(\mathbf{U}) &= [hu, {}^1\!/\!_2 u^2 + gh]^T, \\ \mathbf{G} &= [0, -g\frac{\partial H}{\partial x}]^T. \end{aligned} \tag{13}$$

Applying the absorbing boundary condition machinery described we have

$$\mathbf{H} = (u_0 - c_0) \begin{bmatrix} 1 & -h_0/c_0 \\ -c_0/h_0 & 1 \end{bmatrix}, \quad h_0, u_0 = \textbf{reference state}, c_0 = \sqrt{gh_0}. \tag{14}$$

Applying shallow water AL's to free surface NS

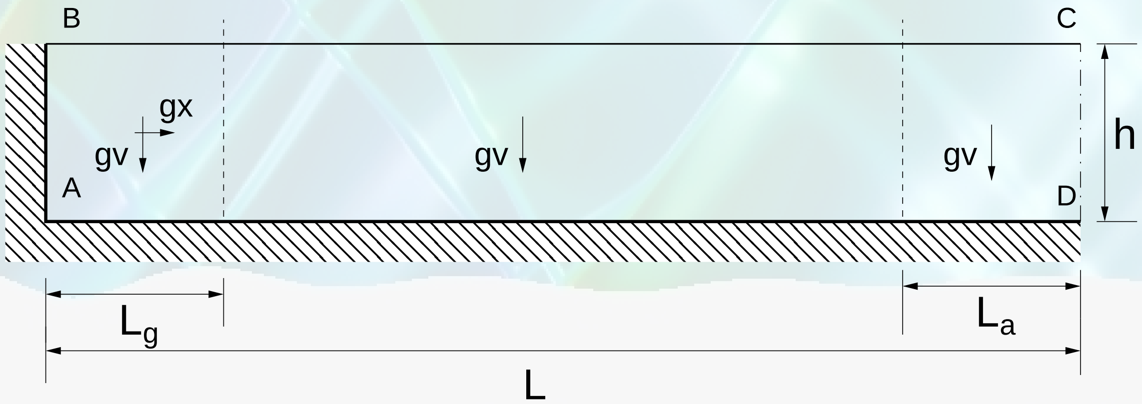
Now we can try to apply these AL to Navier-Stokes eqs. with free surface. The problem is that we have not a h variable. We could determine h by determining a local free surface position. But it's better to replace h from an expression for free waves. In potential theory free surface waves have the following expression

$$\phi(x,z,t) = A\exp\{\mathrm{i}(kx-\omega t)\}\frac{\cosh(kz)}{\cosh(kh_0)}, \quad \mathbf{u} = \nabla\phi,$$

$$\text{so: } g(h-h_0) = \left(\frac{p-p_{\rm atm}}{\rho} + gz\right)\frac{\cosh(kz)}{\cosh(kh_0)}, \qquad (15)$$

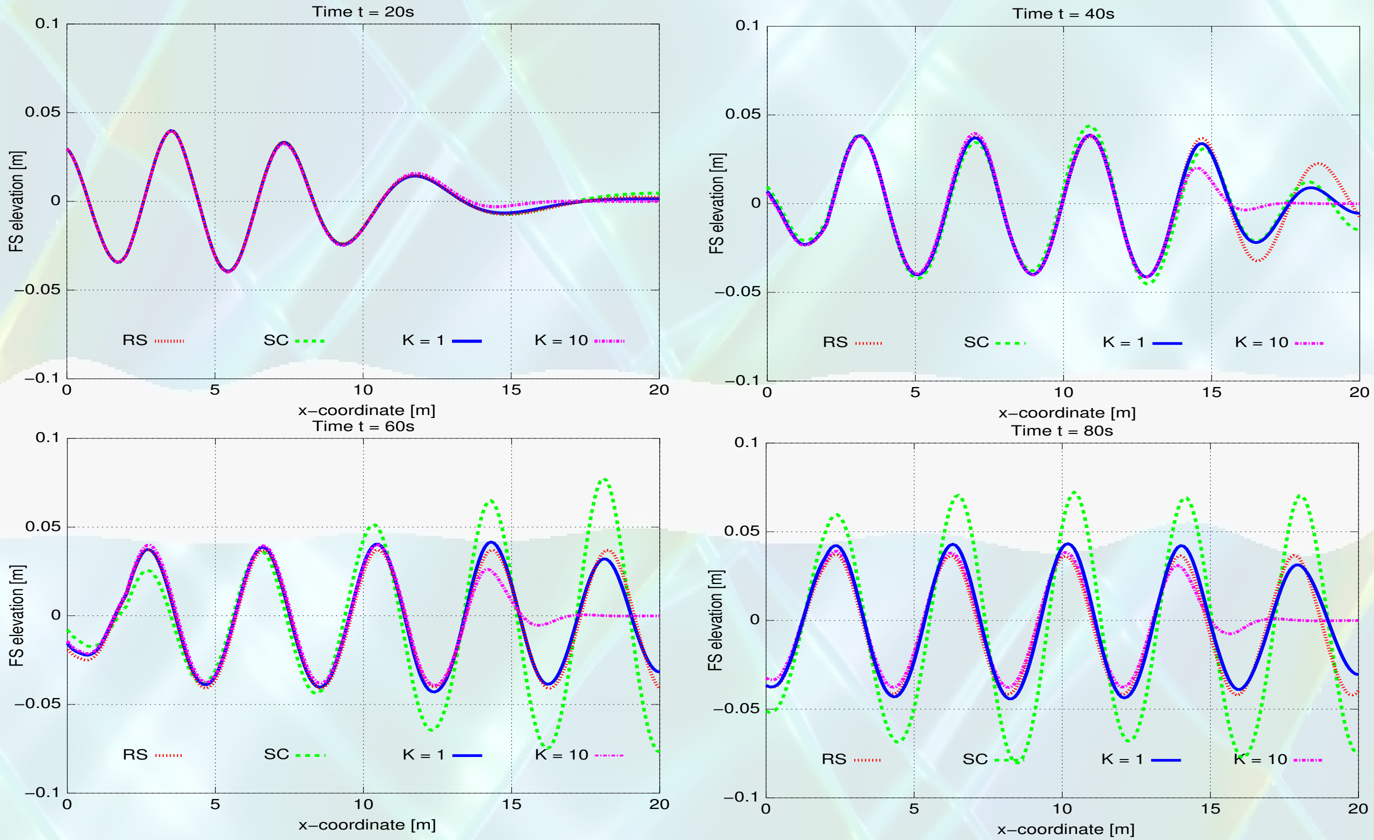
$$g(h-h_0) \approx \frac{p-p_{\rm atm}}{\rho} + gz, \quad (\text{if } kz \ll 1, \text{SW waves}).$$

Example: Open channel with waves generator



- $L = 20\ \mathrm{m}$, mean water depth $h = 1\ \mathrm{m}$, $\nu = 10^{-5}\ \mathrm{m^2 s^{-1}}$
- Wave generator at $L_g = 1.9365\ \mathrm{m}$. $g_x = A\, g_v\, \sin(\omega t)$ ($\omega = 6.28\ \mathrm{m}$, $\lambda = 3.873\ \mathrm{Hz}$.
- Slip boundary at BAD. CD=absorbing.
- Free surface is dealt with surface tracking and ALE movement of mesh (Battaglia et.al. IJCFD 24(3) 121-133 (2010)).
- NS solved with SUPG+PSPG (Tezduyar et.al. CMAME 95(2)

Example: Open channel with waves generator (cont.)



Free surface elevation for the reference solution (RS), the solution in the short domain with slip condition at the outlet (SC), and the last two solved with absorbing boundary conditions (ABC).

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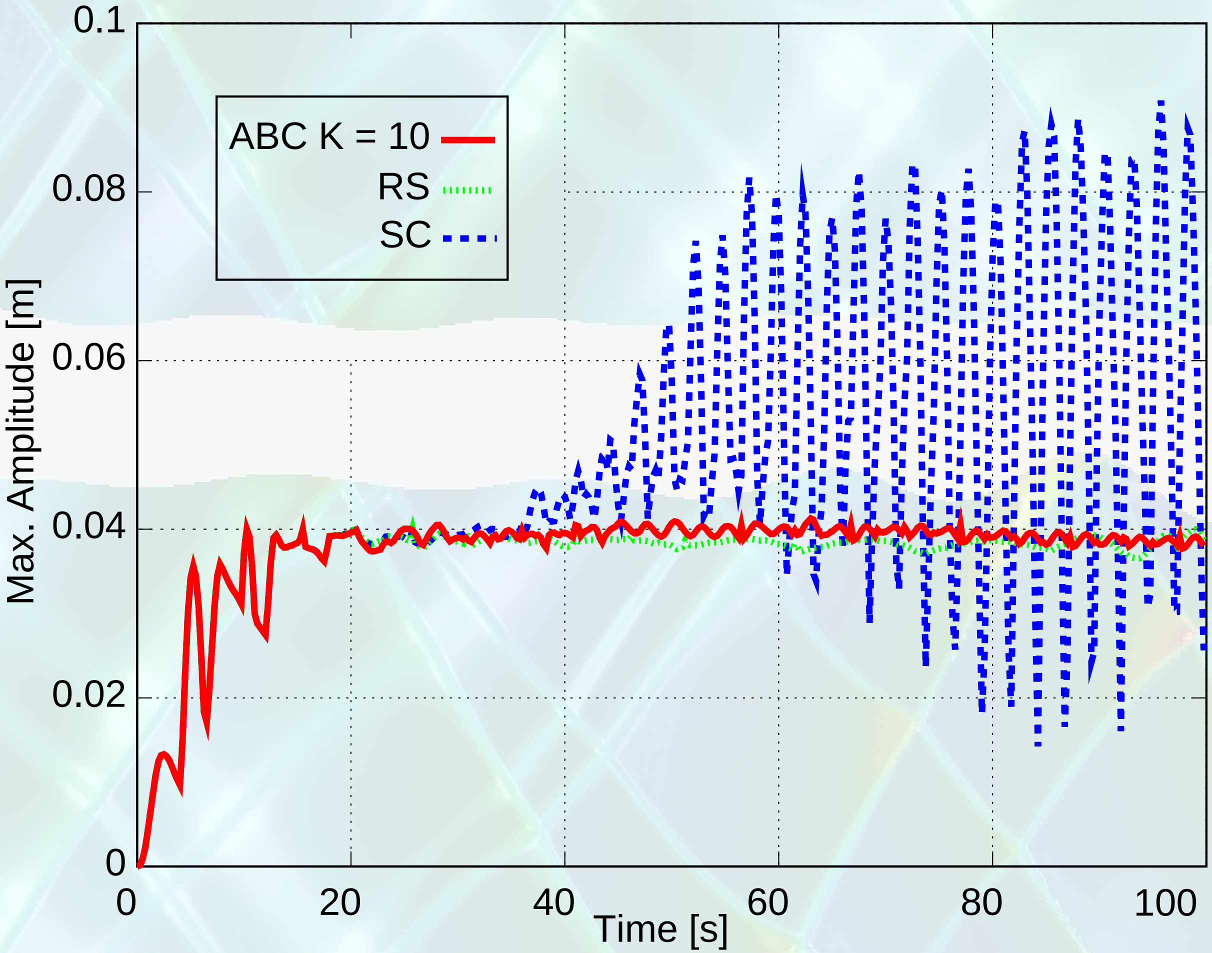
Example: Open channel with waves generator (cont.)

Maximum amplitude in each time step for the reference solution (RS), the solution in the short domain with slip condition at the outlet (SC), and the last one solved with absorbing boundary conditions (ABC) with a penalization of $K = 10$**.**



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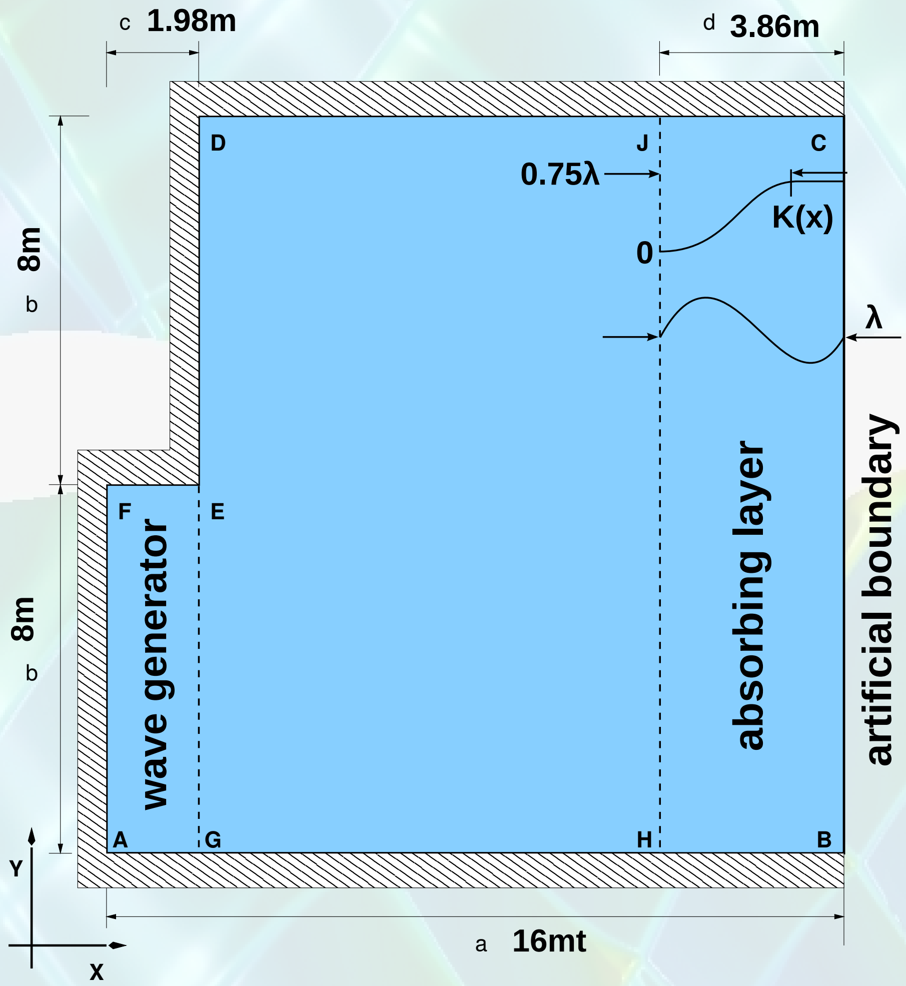
Rectangular domain with waves generator



- $h_0 = 8$ m, $g = 1 \mathrm{m/s^2}$, $\nu = 10^{-4} \mathrm{m^2/s}$
- **wave generator at entry ($\lambda/2$):** $g_x = A_c\, g_v\, \sin(\omega t)$, $\omega = 6.28 \mathrm{s^{-1}}$, $A_c = 0.06$.
- $K = 10$ *(launch video absocond)*

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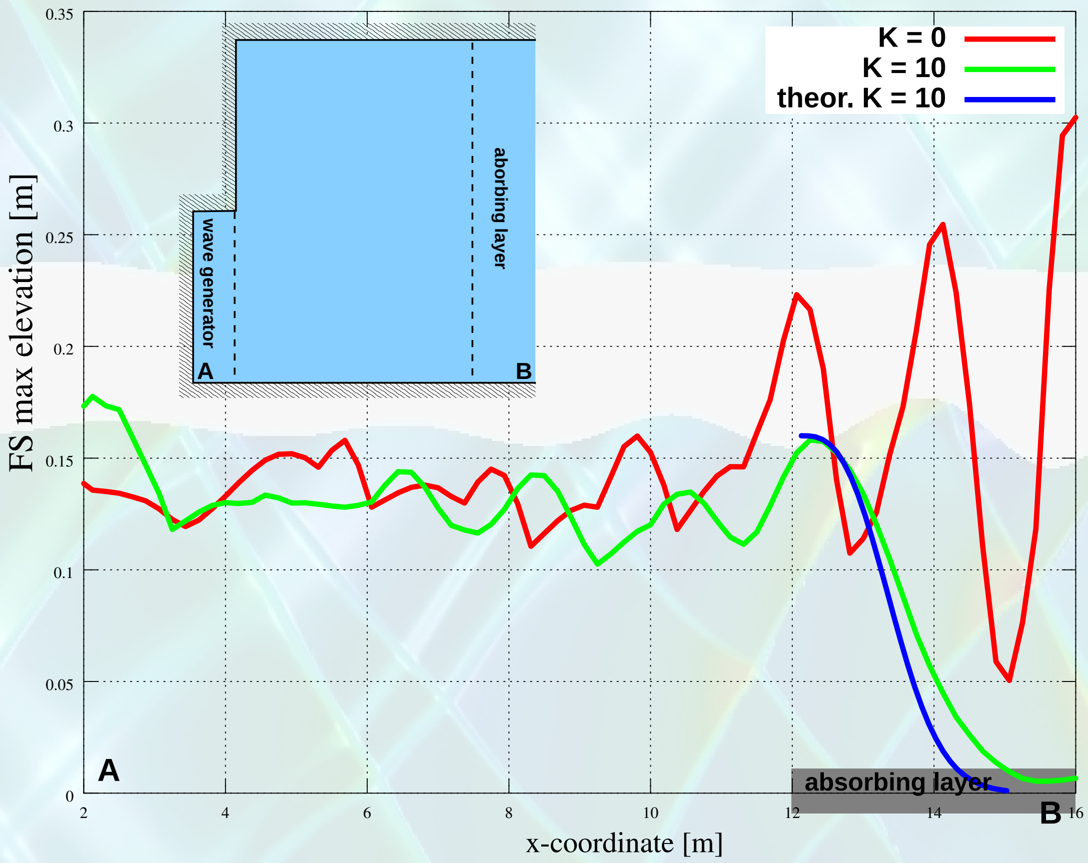
Rectangular domain with waves generator (cont.)

- **Computations diverge without AL ($K = 0$)**
- **Some ripple is found with $K = 10$ (AL is first order only).**
- **Good coincidence between theoretical and numerical max. amplitude in the absorbing layer**

$$A \propto \exp^{-\int K(x)\,\mathrm{d}x}, \quad (16)$$



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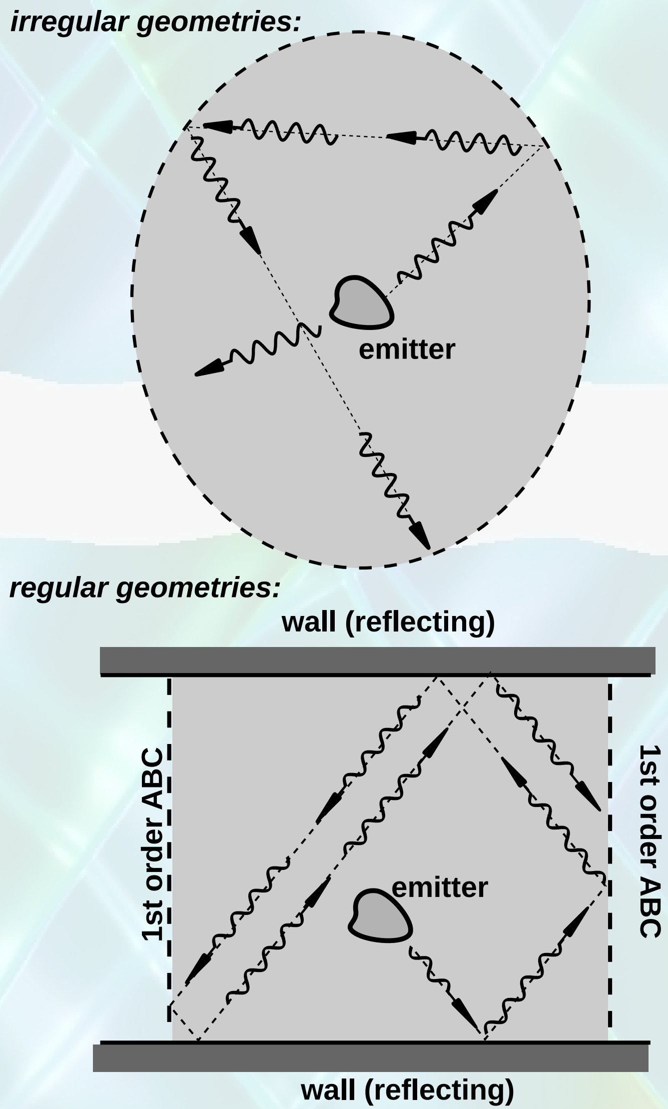
Higher order absorbing boundary conditions



In 2D or 3D a first order ABC will be partially reflecting for non-normal incidence. This can lead to large errors, specially for very regular geometries like a channel.

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Higher order absorbing boundary conditions (cont.)

There are several approaches for higher order aborbing layers, among them a popular approach is the **Perfectly Matched Layer** (PML, J-P Berenger, J. Comput. Phys. 114, 185 (1994)). However,

- It is **addhoc** for each physical problem.
- It is formulated only on **rectangular domains**.
- It's **not very robust**.
- It requires **additional variables** to be defined in the program in the absorbing layer.

The first is the most concerning us. We want an absorbing numerical device with the following characteristics

- To be **automatically computable** from the flux function (and perhaps the Jacobian of the fluxes).
- To be adjustable **higher order** (more probably first and second order).
- To be **robust**.

- **If auxiliary variables are needed, they must be easily computable in the actual context.**

Advective/diffusive systems implemented

- Navier-Stokes compressible flow (☺),
- Shallow water equations (☺),
- Stratified shallow water equations (☺),
- 1D shallow water equations in channels of arbitrary section (☺),
- Scalar advection/diffusion (☺),
- Scalar wave equation (☺)
- Maxwell equations (☹)

(☺ = implemented in PETSc-FEM)

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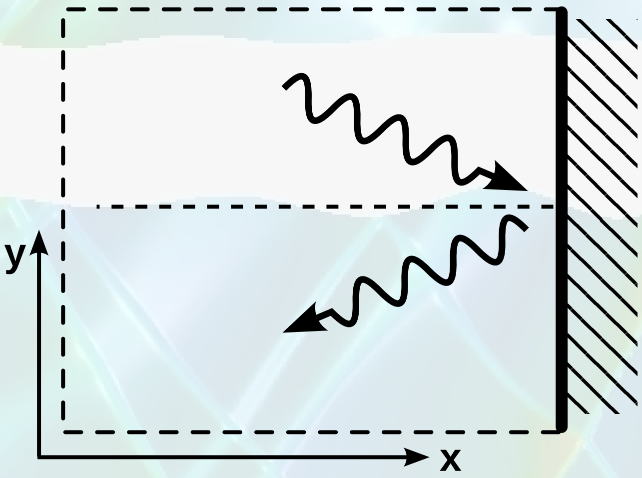
A second order absorbing layer

Assuming that we want to add an absorbing layer in $0 \leq x \leq L$, then the equation would be

$$\mathbf{C}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}\mathbf{U} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} = 0, \tag{17}$$



where $\mathbf{H}$ is the **matrix of absorbing coefficients** to be yet defined.

A second order absorbing layer (cont.)

We now transform Fourier in t and y with associated variables $\mathrm{i}\omega$ and $\mathrm{i}k_y$, i.e. we assume

$$\mathbf{U}(x, y, t) = \hat{\mathbf{U}}(x) \exp\{\mathrm{i}(k_y y - \omega t)\}. \tag{18}$$

Then, we get (for simplicity we drop the hat symbol ($\hat{\mathbf{U}} \to \mathbf{U}$),

$$\begin{aligned} \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + (-\mathrm{i}\omega\mathbf{C} + \mathbf{H} + \mathrm{i}k_y\mathbf{B})\mathbf{U} = 0, \\ \Longrightarrow \quad \frac{\partial \mathbf{U}}{\partial x} + (\mathbf{A}^{-1}\mathbf{H} - \mathrm{i}\omega\mathbf{M}(z))\mathbf{U} = 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} z = k_y/\omega, \\ \mathbf{M}(z) = \mathbf{A}^{-1}(\mathbf{C} - z\mathbf{B}). \end{aligned} \tag{20}$$

A second order absorbing layer (cont.)

$$\frac{\partial \mathbf{U}}{\partial x} + (\mathbf{A}^{-1}\mathbf{H} - \mathrm{i}\omega \mathbf{M}(z))\mathbf{U} = 0, \tag{21}$$

$$z = k_y/\omega, \quad \mathbf{M}(z) = \mathbf{A}^{-1}(\mathbf{C} - z\mathbf{B}). \tag{22}$$

In order to not have reflections $\mathbf{A}^{-1}\mathbf{H}$ must have the same eigenvectors that $\mathbf{M}(z)$. One possibility is then to diagonalize $\mathbf{M}(z)$

$$\mathbf{M}(z) = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}, \tag{23}$$

and then to force $\mathbf{A}^{-1}\mathbf{H}$ to be diagonal in the same basis

$$\mathbf{A}^{-1}\mathbf{H} = \mathbf{Q}\mathbf{\Lambda}_H\mathbf{Q}^{-1}. \tag{24}$$

A second order absorbing layer (cont.)

In order to have an absorbing layer we must have

$$\mathrm{sign}(\lambda(\mathbf{A}^{-1}\mathbf{H})_j) = \mathrm{sign}(\lambda_j), \qquad (25)$$

Transforming Fourier back to t and y we obtain the desired absorbing layer. However, $\mathbf{Q}$ depends on z and so will $\mathbf{H}$ so it would be non-local. In order to have a local operator we perform an expansion of it in powers of z. For instance if we can approximate it to

$$\mathbf{H}(z) \approx \mathbf{H}_0 + z\mathbf{H}_1, \qquad (26)$$

then the absorbing term would be

$$\mathbf{H}_0\mathbf{U} + z\mathbf{H}_1\mathbf{U} = \mathbf{H}_0\mathbf{U} + (\mathrm{i}k_y/\mathrm{i}\omega)\mathbf{H}_1\mathbf{U}. \qquad (27)$$

Transforming Fourier back to (t, y) we get

$$\begin{aligned} \mathbf{C}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{H}(\{\mathbf{U}\}) + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} &= 0, \\ \mathbf{H}(\{\mathbf{U}\}) = \mathbf{H}_0\mathbf{U} + \mathbf{H}_1 \int_{t=0}^{t} \frac{\partial \mathbf{U}}{\partial y}\,\mathrm{d}t. \end{aligned} \tag{28}$$

The simplest choice for (25) is $\lambda(\mathbf{A}^{-1}\mathbf{H})_j = K\lambda_j$, and we get $\mathbf{H}_0 = K\mathbf{C}$, $\mathbf{H}_1 = -K\mathbf{B}$, so that the absorbing layer is

$$\mathbf{C}\frac{\partial \mathbf{U}}{\partial t} + K\left\{\mathbf{C}\mathbf{U} - \mathbf{B}\int_{t=0}^{t} \frac{\partial \mathbf{U}}{\partial y}\,\mathrm{d}t\right\} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{U}}{\partial y} = 0, \tag{29}$$

Note that in the first order case we get a **local operator**.

Numerical evaluation of the absorbing operator

- It involves an **integral over time**, that involves storing an auxiliary variable say $\mathbf{W}$ and then updating with

$$\begin{aligned} \mathbf{W} &= \int_{t=0}^{t} \frac{\partial \mathbf{U}}{\partial y}\,\mathrm{d}t, \\ \frac{(3\mathbf{W}^{n+1} - 4\mathbf{W}^{n} + \mathbf{W}^{n-1})_{jk}}{2\Delta t} &= \frac{(\mathbf{U}_{j,k+1} - \mathbf{U}_{j,k-1})^{n+1}}{2\Delta y}. \end{aligned} \quad (30)$$

- The derivative with respect to y is computed by standard finite difference approximations ($j(k)$ indices is along $x(y)$ axis).
- The operator can be easily evaluated in the context of an **unstructured grid solver** (FEM for instance).
- $K[=]\mathrm{T}^{-1}$ but if a velocity scale $\bar{v}$ is available then we get a **length scale** $\bar{L}_{\mathrm{abso}} = \bar{v}/K$

Examples. The scalar wave equation

The standard representation of the scalar wave equation is

$$\phi_{tt} = c^2 \Delta\phi \tag{31}$$

In 2D we can put it in the form of a first order advective system as

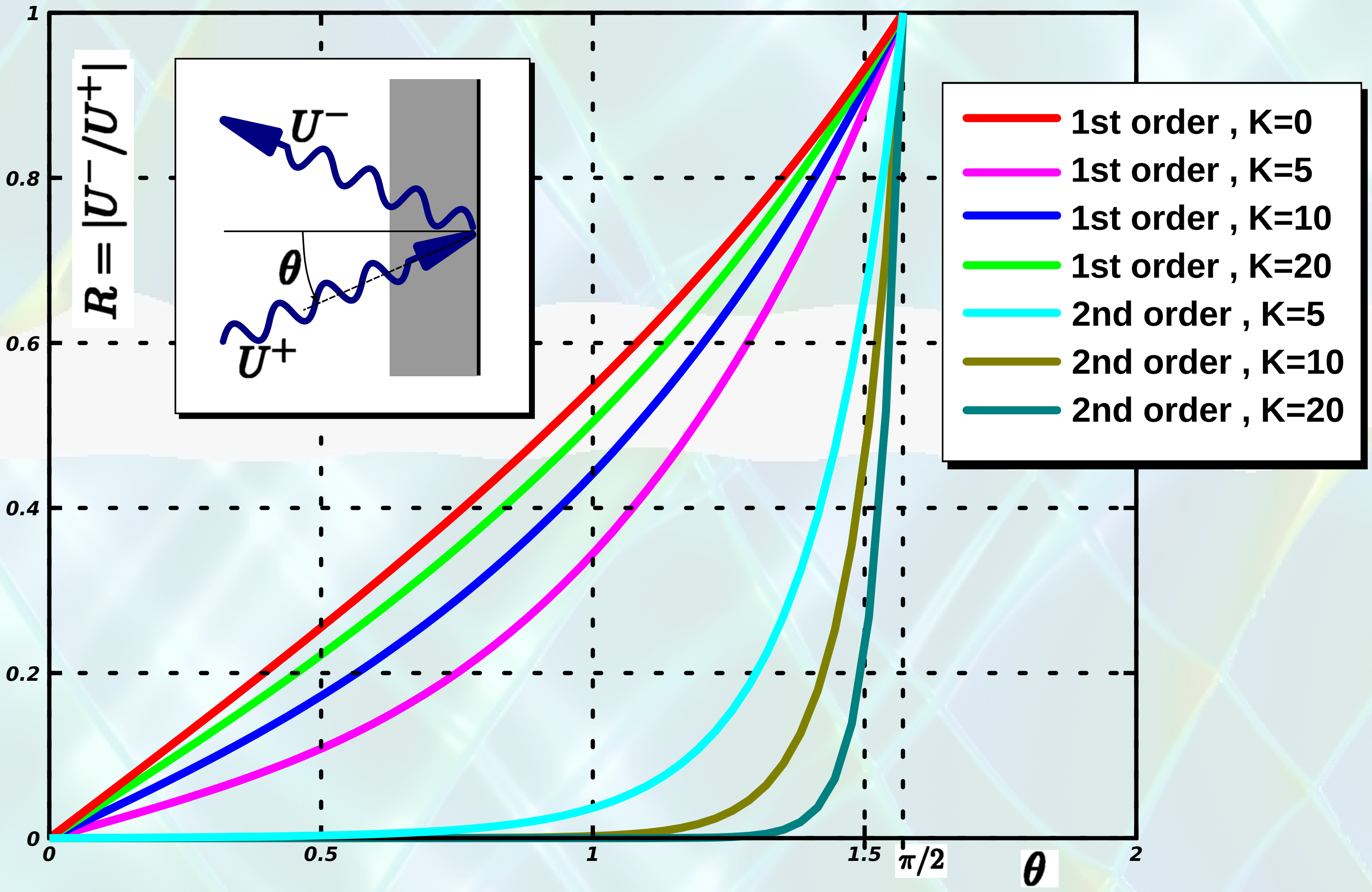
$$\begin{aligned} u_t + c(u_x + v_y) &= 0, \\ v_t + c(-v_x + u_y) &= 0. \end{aligned} \tag{32}$$

and the corresponding vectorial form is

$$\mathbf{U} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A_x = c \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_y = c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{33}$$

u and v satisfy the wave equation, and the dispersion relation is $\omega = ck$ (the same as for the wave equation).

Reflection coefficients



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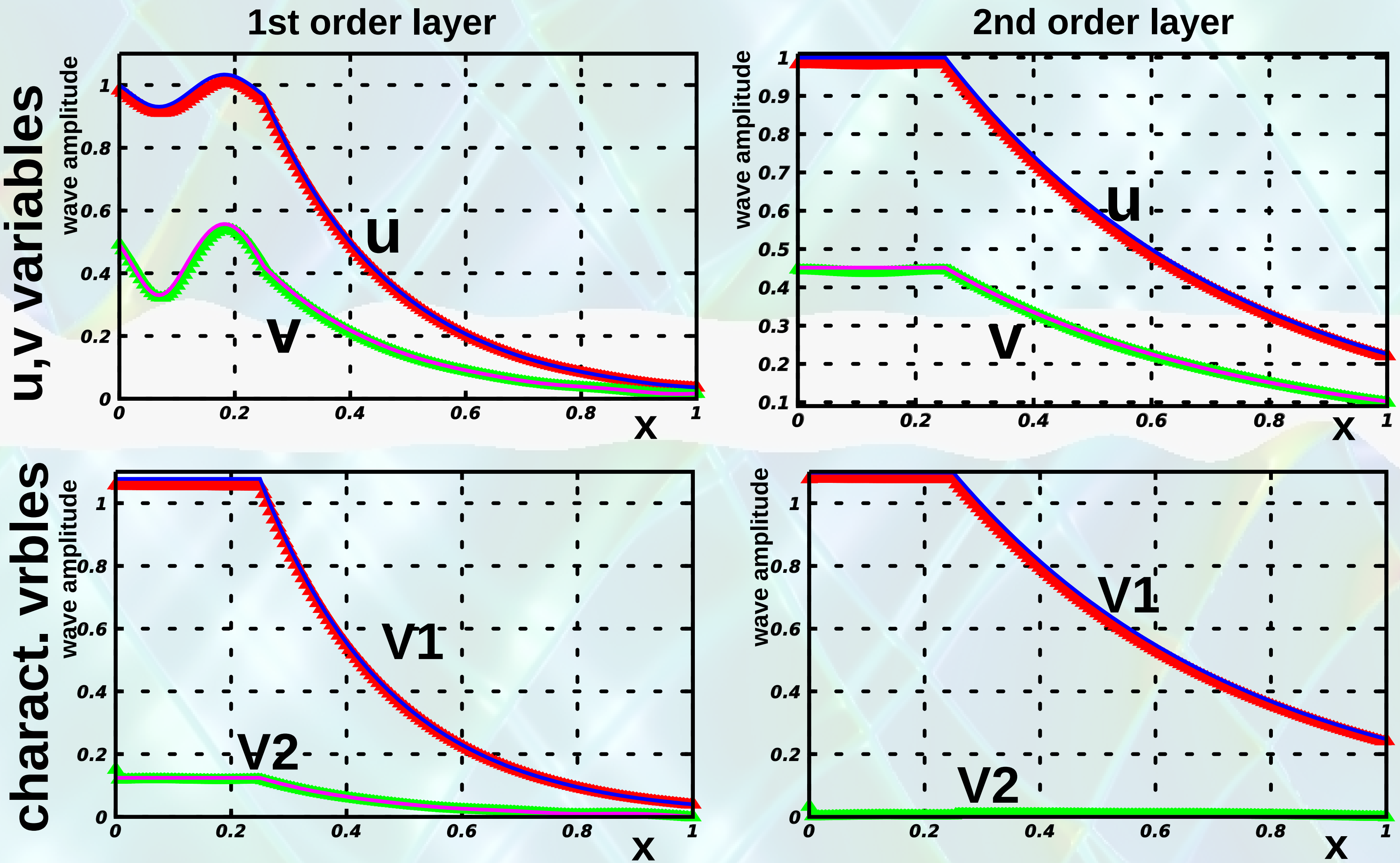
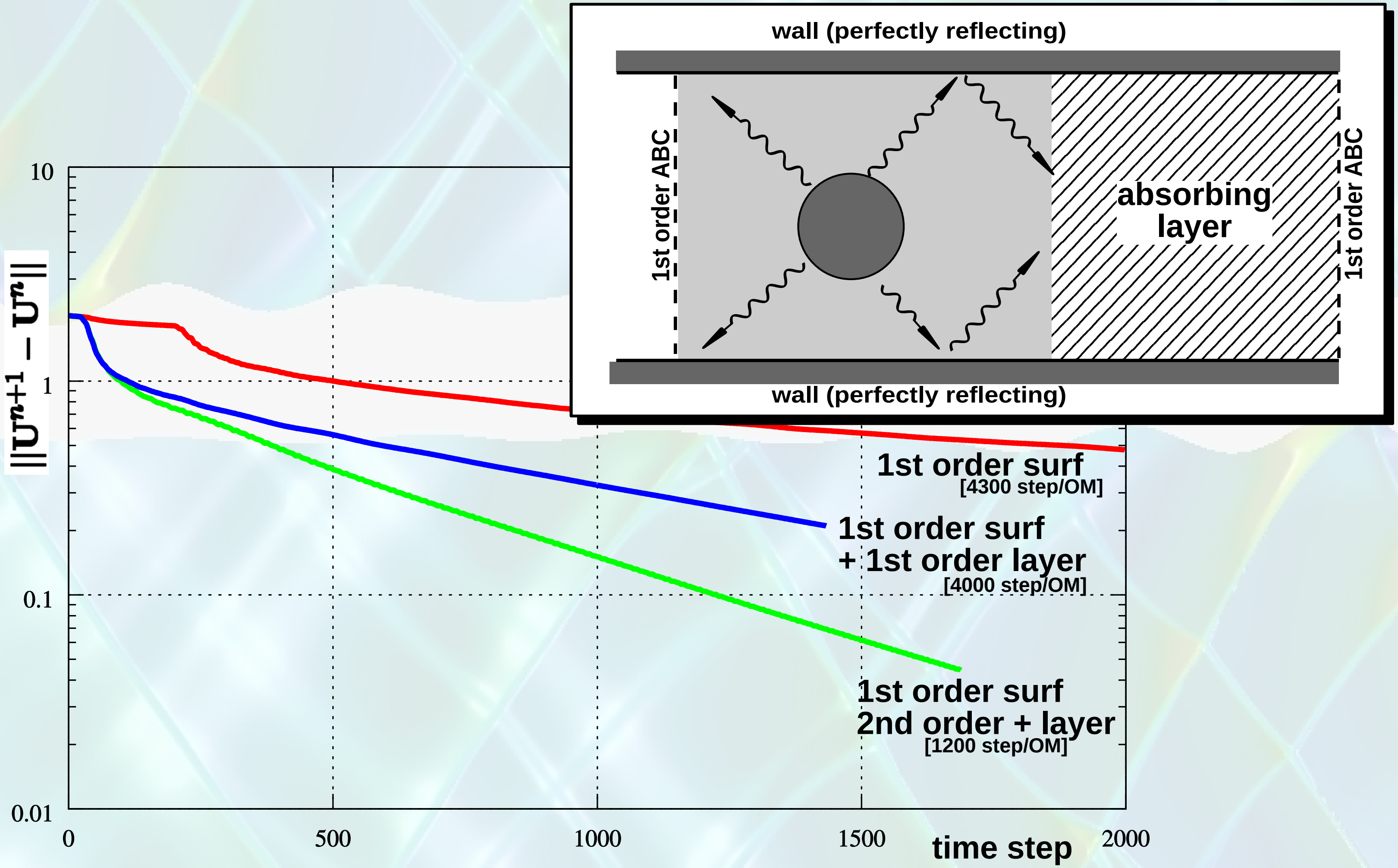
Reflection coefficients (cont.)



Decaying perturbation example



Other choices for the absorbing matrix coefficient

Recall that the requirement on H is

$$\mathrm{sign}(\lambda(\mathbf{A}^{-1}\mathbf{H})_j) = \mathrm{sign}(\lambda_j), \tag{34}$$

some possible choices are

$$\begin{aligned} \lambda(\mathbf{A}^{-1}\mathbf{H})_j) = \lambda_j & \quad\Longrightarrow\quad \mathbf{H}_0 = \mathbf{C}, \\ \lambda(\mathbf{A}^{-1}\mathbf{H})_j) = \mathrm{sign}(\lambda_j) & \quad\Longrightarrow\quad \mathbf{H}_0 = \mathbf{A}|\mathbf{A}^{-1}\mathbf{C}|, \\ \lambda(\mathbf{A}^{-1}\mathbf{H})_j) = 1/\lambda_j & \quad\Longrightarrow\quad \mathbf{H}_0 = \mathbf{A}|\mathbf{C}^{-1}\mathbf{A}|, \end{aligned} \tag{35}$$

The expansion $\mathrm{H}(z) \approx \mathrm{H}_0 + z\mathrm{H}_1$ can be done numerically if it is not possible to do analytically, i.e. compute $\mathrm{H}(z)$ for several z and fit with polynomials for each entry in H.

Conclusions and future work

- The presented scheme is a first order AL for the NS equations with free surface.
- Good absorption is observed with absorbing layer of 1λ or less.
- Extension to higher order AL’s or ABC’s is an ongoing work.
- The AL has been shown for a tracking surface ALE formulation, but can be extended to Level-Set or VOF schemes as well.

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