COMPLETENESS OF MUCKENHOUPT CLASSES

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Abstract. In this note we prove that the Hausdorff distance between compact sets and the Kantorovich distance between measures, provide an adequate setting for the convergence of Muckenhoupt weights. The results which we prove on compact metric spaces with finite metric dimension can be applied to classical fractals.

1. Introduction

In this section we shall introduce the problem considered in this note through a concrete example on the most classical fractal: the Cantor set. We shall also state here the main result of this note.

Let us start by introducing some basic notation. Even when our general result contained in Theorem 1.2 holds in quasi-metric spaces, the general theory of spaces of homogeneous type developed by Macías and Segovia in [12], allows us to reduce our environment to a somehow simpler situation. In this note (X, d) is a fixed given compact metric space. Without loosing generality we assume that the d-diameter of X is less than one. We shall also assume that (X, d) has finite metric (or Assouad) dimension. This means that there exists a constant N such that no d-ball in X with radius $r > 0$ contains more than N points of any $\frac{r}{2}$-disperse subset of X. It is well known, and perhaps the most important result of the theory of spaces of homogeneous type, that the finiteness of the metric dimension is equivalent to the existence of a doubling measure on the Borel subsets of X (see [15] and [16]).

Let us by start by a brief introduction of a distance on the family of all closed probabilistic subspaces (Y, d, μ) of (X, d) is such a way that a sequence $(Y_n, d\mu_n)$ converges to (Y, d, μ) in that distance if and only if Y_n tends to Y in the Hausdorff sense and μ_n tends to μ in the weak star sense. This can be accomplished by adding the Hausdorff distance between compact sets and the Kantorovich distance between measures. We shall borrow from [4] the notation and basic results which we briefly introduce for the sake of completeness.

Let $\mathcal{K} = \{K \subseteq X : K \neq \emptyset,\ K \text{ compact}\}$. With $[A]_\varepsilon$ we shall denote the ε-enlargement of the set $A \subset X$; i.e. $[A]_\varepsilon = \bigcup_{x\in A} B_d(x, \varepsilon) = \{y \in X : d(y, A) < \varepsilon\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ and $d(x, A) = \inf\{d(x, y) : y \in A\}$. Given A and E two sets in $\mathcal{K}$ the Hausdorff distance from A to E is given by

$$d_H(A, E) = \inf\{\varepsilon > 0 : A \subseteq [E]_\varepsilon \text{ and } E \subseteq [A]_\varepsilon\}.$$

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It is well known that $(\mathcal{K}, d_H)$ is a complete metric space (see [10]).

On the other hand, the Kantorovich-Rubinstein distance on the set $\mathcal{P}(X)$ of all the positive Borel probabilities μ ($\mu(X) = 1$) is is defined as follows

$$d_K(\mu, \nu) = \sup\left\{\left|\int f\, d\mu - \int f\, d\nu\right| : f \in \mathrm{Lip}_1(X)\right\},$$

where $\mathrm{Lip}_\Lambda(X)$ means that $|f(x) - f(y)| \leq \Lambda\, d(x, y)$ for every x and $y \in X$. The metric space $(\mathcal{P}(X), d_K)$ is complete, and the d_K-convergence of a sequence is equivalent to its weak star convergence to the same limit. We shall also use the notation $\mathrm{Lip}(X) := \bigcup_{\Lambda>0} \mathrm{Lip}_\Lambda(X)$.

Set $\mathscr{X} = \mathcal{K} \times \mathcal{P}$, and given two elements (Y_i, μ_i) of $\mathscr{X}$, $i = 1, 2$, we define

$$d_{HK}((Y_1, \mu_1), (Y_2, \mu_2)) = d_H(Y_1, Y_2) + d_K(\mu_1, \mu_2),$$

so that $(\mathscr{X}, d_{HK})$ becomes a complete metric space. Let

$$\mathcal{E} = \{(Y, \mu) \in \mathscr{X} : \operatorname{supp} \mu \subseteq Y\},$$

where $\operatorname{supp} \mu$ denotes the support of μ, i.e. the complementary of the largest open set G in X for which $\int \varphi\, d\mu = 0$ for every $\varphi \in \mathcal{C}(X)$, the space of all continuous real valued functions on X, with $\operatorname{supp} \varphi \subseteq G$, and $\operatorname{supp} \varphi$ is the closure of the set $\{\varphi \neq 0\}$. We have that $(\mathcal{E}, d_{HK})$ is a complete metric subspace of $(\mathscr{X}, d_{HK})$.

Let A be a given real number with $A \geq 1$. Let $\mathcal{D}(A)$ be the set of all couples (Y, μ) in $\mathcal{E}$ such that the inequalities

$$0 < \mu(B_d(y, 2r)) \leq A\mu(B_d(y, r))$$

hold for every $y \in Y$ and every $r > 0$. Such a couple (Y, μ) is usually called a space of homogeneous type if we understand that the metric is the restriction of d to Y. In [4] we prove the following elementary completeness type result for the doubling condition. If $\{(Y_n, \mu_n) : n \in \mathbb{N}\}$ is a sequence in $\mathcal{D}(A)$ and $(Y_n, \mu_n) \xrightarrow{d_{HK}} (Y, \mu)$, then $(Y, \mu) \in \mathcal{D}(A^4)$.

We shall introduce the Muckenhoupt classes on a couple $(Y, \mu) \in \mathcal{D}(A)$. Given $1 < p < \infty$ and a couple $(Y, \mu) \in \mathcal{D}(A)$, we say that a non negative, non trivial and locally integrable function w on Y is a weight on (Y, μ). We shall say that a weight w is an $A_p = A_p(Y, \mu)$ Muckenhoupt weight if there exists a constant C such that the inequality

$$\left(\int_B w\, d\mu\right)\left(\int_B w^{-\frac{1}{p-1}}\, d\mu\right)^{p-1} \leq C\, (\mu(B))^p$$

holds for every d-ball B in Y. We shall also use the notation $w \in A_p(Y, \mu)$, and we shall say that C is a Muckenhoupt constant for w. A classical reference for the theory of Muckenhoupt weights in the Euclidean space is the book by José García Cuerva and José Luis Rubio de Francia (see [11]).

Let us take a look at the Cantor set and its standard approximations in the setting described in the preceding general framework. The natural environment for the special case of the Cantor set is the space $X = [0,1]$ with the usual distance. Let us write

$$C = \bigcap_{n=1}^{\infty} C_n, \quad C_n = \bigcup_{j=1}^{2^n} I_n^j, \quad I_n^j = [a_n^j, b_n^j]\,,$$

where C_n is the n-th step in the construction of the Cantor set. For each positive integer n, set $Y_n = \{b_n^j : j = 1,2,\ldots,2^n\}$, in other words, Y_n is the collection of all the right points of each interval in C_n. Let μ_n be the discrete measure defined on Y_n by $\mu_n(\{x\}) = 2^{-n}$ for each $x \in Y_n$. Let us notice that Y_n can be obtained by dividing by 3^n all the non-negative integers whose expansion in basis 3 do not contain the digit 2 and having at most n digits. So that each point $x \in Y_n$ can be identified with an n-tuple $(x_1, x_2, \ldots, x_n)$ where each x_i is zero or one. With this notation, following [5], define $d_n : Y_n \times Y_n \to \mathbb{R}^+ \cup \{0\}$ by

$$d_n(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 3^{-j}, & \text{if } x_i = y_i \text{ for every } i < j \text{ and } x_j \neq y_j. \end{cases}$$

It is easy to see that d_n is a distance on Y_n. Notice that for $x \in Y_n$ and j a positive integer, we have

$$B_{d_n}\left(x, 3^{-j}\right) := \{y \in Y_n : d_n(x,y) < 3^{-j}\} = \{y \in Y_n : y_i = x_i,\ i = 1,2,\ldots,j\}\,,$$

hence

$$\operatorname{card}\left(B_{d_n}\left(x,3^{-j}\right)\right) = \begin{cases} 2^{n-j}, & j \le n, \\ 1, & j \ge n. \end{cases}$$

So that

$$\mu_n\left(B_{d_n}\left(x,3^{-j}\right)\right) = \begin{cases} 2^{-j}, & j \le n, \\ 2^{-n}, & j \ge n. \end{cases}$$

Observe that given a positive integer n and $x, y \in Y_n$, $x \neq y$, with $d_n(x,y) = 3^{-j}$, we necessarily have that

$$x - y = \sum_{i=j}^{n} 3^{-i}(x_i - y_i),$$

from which we obtain the inequalities

$$d_n(x,y) \le |x-y| \le 3 d_n(x,y),$$

for every n and every $x, y \in Y_n$. Then, if $B(x,r)$ is the interval of length $2r$ centered at x, we have

$$B(x,r) \cap Y_n \subseteq B_{d_n}(x,r) \subseteq B(x,3r) \cap Y_n,$$

for every n, every $x \in Y_n$ and every $r > 0$.

Then each (Y_n, μ_n) belongs to $\mathcal{D}(A)$ with respecto to the usual distance for $A = 4^3$. Also $(Y_n,\mu_n) \xrightarrow{d_{HK}} (C,\mu)$, where μ is the Hausdorff measure of dimension $s = \log 2/\log 3$ on the Cantor set C.

Since the set Y_n is finite and the measure μ_n is essentially counting, the basic facts of harmonic analysis on the space of homogeneous type $(Y_n, |\cdot|, \mu_n)$ are somehow trivial. In particular any positive function defined on Y_n becomes a Muckenhoupt weight belonging to every $A_p(Y_n, |\cdot|, \mu_n)$ class. So that interesting problems arise only trying to obtain uniform bounds.

We start by searching for the possible values of $\alpha \in \mathbb{R}$ for which the functions $w_n(y) := |y|^\alpha = y^\alpha$, $y \in Y_n$, are Muckenhoupt A_p weights on $(Y_n, |\cdot|, \mu_n)$ *uniformly* in n. Then, after a normalization to a probability measure of $w_n d\mu_n$ on Y_n, we look at its weak limit μ supported on the Cantor set C. Doing this we recover for fractional dimension the classical Euclidean fact: $w(x) = |x|^\alpha$ belongs to $A_p(\mathbb{R}^n)$ if and only if $-n < \alpha < n(p-1)$ $(1 < p < \infty)$.

For this sequence of weights $w_n = y^\alpha$ on the sequence of spaces of homogeneous type $(Y_n, |\cdot|, \mu_n)$ we have the desired uniform A_p condition for an adequate interval of values for α.

Proposition 1.1. *For $s = \frac{\log 2}{\log 3}$ and $-s < \alpha < s(p-1)$ there exists a constant $C = C(\alpha)$ such that $w_n \in A_p(Y_n, |\cdot|, \mu_n)$ with Muckenhoupt constant C for every $n \in \mathbb{N}$.*

Proof. Let us fix α in the open interval $(-s, s(p-1))$. We shall show that there exists a constant C independent of n such that

$$\left(\int_{B(x,r)} w_n \, d\mu_n\right)\left(\int_{B(x,r)} w_n^{-\frac{1}{p-1}} \, d\mu_n\right)^{p-1} \leq C(\mu_n(B(x,r)))^p \tag{1}$$

for every n, every $x \in Y_n$ and every $r > 0$. Both integrals on the left hand side of (1) involve positive or negative powers of the variable $y \in Y_n$. So, let us start by obtaining upper estimates for the integrals of these type of functions on $B(x,r)$.

Notice first that for any $\beta \in \mathbb{R}$ we have that

$$\int_{B(x,r)} y^\beta \, d\mu_n(y) = \frac{x^\beta}{2^n}$$

for every $0 < r \leq 3^{-n}$ and every $x \in Y_n$, since $B(x,r) \cap Y_n = \{x\}$. Let us then assume that $3^{-n} < r \leq 1$, and take an integer j such that $0 \leq j \leq n$ and $3^{-j} < r \leq 3^{1-j}$. Now, as in the Euclidean case, we divide our analysis according to the relative position of the "first point" in Y_n, 3^{-n}, and x with respect to the size r of the given ball. Let us first assume that $x \in Y_n$ and $0 \leq x - 3^{-n} < 2r$. Then, for $\beta \geq 0$ we have

$$\begin{aligned}
\int_{B(x,r)} y^\beta \, d\mu_n &\leq \int_{B(3^{-n},3r)} y^\beta \, d\mu_n(y) \\
&\leq \int_{B_{d_n}(3^{-n},3^{2-j})} y^\beta \, d\mu_n(y) \\
&\leq 3^{(3-j)\beta} \mu_n(B_{d_n}(3^{-n}, 3^{2-j})) \\
&= 3^{(3-j)\beta} 2^3 \mu_n(B_{d_n}(x, 3^{-j-1})) \\
&\leq 3^{(3-j)\beta} 2^3 \mu_n(B(x, 3^{-j})) \\
&\leq 3^{(3-j)\beta} 2^3 \mu_n(B(x, r)).
\end{aligned}$$

On the other hand, for $-s < \beta < 0$ we have

$$\begin{aligned}
\int_{B(x,r)} y^\beta \, d\mu_n(y) &\leq \int_{B_{d_n}(3^{-n},3^{-j+2})} y^\beta \, d\mu_n(y) \\
&= \frac{3^{-n\beta}}{2^n} + \int_{B_{d_n}(3^{-n},3^{-j+2}) - \{3^{-n}\}} (y - 3^{-n})^\beta \, d\mu_n(y) \\
&= \frac{3^{-n\beta}}{2^n} + \sum_{\ell=0}^{n-j+1} \int_{3^{-\ell-j+1} \leq d_n(3^{-n},y) < 3^{-\ell-j+2}} (y - 3^{-n})^\beta \, d\mu_n(y)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{3^{-n\beta}}{2^n} + 3\sum_{\ell=0}^{n-j+1} \int_{3^{-\ell-j+1}\le d_n(3^{-n},y)<3^{-\ell-j+2}} d_n^{\beta}(y,3^{-n})\,d\mu_n(y)\\
&\le \frac{3^{-n\beta}}{2^n} + 3\sum_{\ell=0}^{n-j+1} 3^{\beta(-\ell-j+1)}2^{-\ell-j+2}\\
&\le 3^{\beta(1-j)+1}2^{2-j}\sum_{\ell=0}^{\infty}\left(\frac{3^{-\beta}}{2}\right)^{\ell}\\
&\le 2^4 3^{1+\beta}\frac{1}{2-3^{-\beta}}3^{-\beta j}\mu_n(B(x,r)).
\end{aligned}$$

Then

$$\begin{aligned}
&\left(\int_{B(x,r)} w_n\,d\mu_n\right)\left(\int_{B(x,r)} w_n^{-\frac{1}{p-1}}\,d\mu_n\right)^{p-1}\\
&= \left(\int_{B(x,r)} y^{\alpha}\,d\mu_n\right)\left(\int_{B(x,r)} y^{-\frac{\alpha}{p-1}}\,d\mu_n\right)^{p-1}\\
&\le \begin{cases} \frac{2^{4p-1}3^{\alpha+p-1}}{\left(2-3^{\frac{2\alpha}{p-1}}\right)^{p-1}}\mu_n^p(B(x,r)), & 0<\alpha<s(p-1),\\[2mm] \mu_n^p(B(x,r)), & \alpha=0,\\[2mm] \frac{2^{3p+1}3^{1-2\alpha}}{2-3^{-\alpha}}\mu_n^p(B(x,r)), & -s<\alpha<0.\end{cases}
\end{aligned}$$

In the case $x-3^{-n}\ge 2r$ we immediately have

$$\int_{B(x,r)} y^{\beta}\,d\mu_n \le \begin{cases} (x+r)^{\beta}\mu_n(B(x,r)), & \text{if } \beta\ge 0,\\ (x-r)^{\beta}\mu_n(B(x,r)), & \text{if } \beta<0.\end{cases}$$

Since $x\ge 2r$ we have $\frac{x+r}{x-r}\le 3$. Then

$$\left(\int_{B(x,r)} y^{\alpha}\,d\mu_n\right)\left(\int_{B(x,r)} y^{-\frac{\alpha}{p-1}}\,d\mu_n\right)^{p-1} \le 3^{|\alpha|}(\mu_n(B(x,r)))^p.$$

□

Proposition 1.1 shows that the sequence of weights w_n is uniformly in A_p of the corresponding domain and measure. The question considered in this note is whether or not from this uniform property on the approximating sequence it is possible to deduce the Muckenhoupt condition for the limit weight on the limit measure space. The main aim of this paper is to prove a general result in this direction which is essentially contained in the next statement. A quantitative more precise version of Theorem 1.2 is contained in Theorem 3.2.

Theorem 1.2. *Let $1<p<\infty$ be given. Let $\{(Y_n,\mu_n): n\in\mathbb{N}\}$ be a given sequence in $\mathcal{D}(A)$ such that $(Y_n,\mu_n)\xrightarrow{d_{HK}}(Y,\mu)$. Let $\{w_n: n\in\mathbb{N}\}$ a sequence of weights for which there exists a fixed constant C such that $w_n\in A_p(Y_n,\mu_n)$ with Muckenhoupt constant C for each n, normalized in such a way that $\int w_n\,d\mu_n$=1 for each n. If $w_n d\mu_n$ converges to $d\nu$ in the weak star sense, then there exists a weight w on Y such that $d\nu = w d\mu$, and $w\in A_p(Y,\mu)$.*

Notice that from Theorem 1.2 and Proposition 1.1 we may conclude that for $s = \frac{\log 2}{\log 3}$ and $-s < \alpha < s(p-1)$, the weight $w(y) = y^\alpha$ belongs to A_p on the Cantor set C with its natural Hausdorff measure μ. Let us point out that for this very special example, since from the results in [13] the limit Cantor space $(C, |\cdot|, \mu)$ is an s-normal space of homogeneous type, it is possible to show that the only powers of the distance to a fixed point in C which are A_p weights are those in the interval $(-s, s(p-1))$ (see [1]).

The key of our argument is to give an equivalent version of the A_p condition using smooth mean values of Lipschitz functions instead of maximal operators, in order to be able to apply the metric Kantorovich view of the weak star convergence of measures.

In Section 2 we review some basic facts of Muckenhoupt's theory on A_p-weights and we obtain some reformulations of the A_p condition which are suitable to obtain the proof of Theorem 1.2, which is given in Section 3.

2. Basic Muckenhoupt theory

The first basic result of the Muckenhoupt theory is the equivalence of the A_p condition with the boundedness of the Hardy-Littlewood maximal operator on the corresponding weighted L^p space. We shall state this equivalence in two separate theorems for the general setting. The first one follows exactly the lines of the Euclidean case (see [9] for example). The second can be found in [2] and is an extension to spaces of homogeneous type of the technique in [8].

Theorem 2.1. *Let (Y, d, μ) be a space of homogeneous type. Let $1 < p < \infty$ and let w be a weight in Y such that the Hardy-Littlewood maximal operator satisfies the inequality*

$$\int |Mf|^p \, w \, d\mu \leq C \int |f|^p w \, d\mu$$

for some constant C and every $f \in L^1_{loc}$. Then $w \in A_p(Y, d, \mu)$.

Theorem 2.2. *Let (Y, d, μ) be a space of homogeneous type and let $w \in A_p$ for some $1 < p < \infty$. Then the Hardy-Littlewood maximal operator is bounded in $L^p(w)$. In other words*

$$\int |Mf|^p \, w \, d\mu \leq C \int |f|^p w \, d\mu$$

for every function $f \in L^1_{loc}$, where C depends only on the geometric constants, on p and on the Muckenhoupt constant for w.

In both results M is the non centered Hardy-Littlewood maximal function defined by taking mean values with respecto to μ over the family of d-balls on Y.

The boundedness on $L^p(\nu)$ of the Hardy-Littlewood maximal function defined with respecto to the measure μ...

Theorem 2.3. *Let (Y, d) be a compact metric space and let (Y, d, μ) be a space of homogeneous type. Let ν be a Borel measure on Y which is positive and finite on* *each d-ball of Y. If there exist $1 < p < \infty$ and $C > 0$ such that the inequality*

$$\int_Y \left(\frac{1}{\mu(B_d(x,r))} \int_{B_d(x,r)} |f(y)|\, d\mu(y) \right)^p d\nu(x) \leq C \int_Y |f(y)|^p\, d\nu(y) \tag{2}$$

holds for every $r > 0$ and every $f \in L^1(Y,\mu)$, then ν is a doubling measure on Y, ν is absolutely continuous with respect to μ, and the Radon-Nikodyn derivative of ν with respect to μ is an $A_p(Y,d,\mu)$ weight.

Notice that the boundedness of the Hardy-Littlewood maximal operator with respect to ν implies (2), hence the results of Theorem 2.3 are valid.

In the proof of Theorem 2.3 we shall use a powerful tool of real analysis which has been constructed on spaces of homogeneous type by M. Christ in [7]: the "dyadic cubes". Given $(Y,\mu) \in \mathcal{D}(A)$ for some $A \geq 1$, for every $j \in \mathbb{N} \cup \{0\}$ there exists a finite initial interval $\mathcal{K}(j)$ such that for each $(j,k) \in \mathcal{A} := \{(j,k) : j \in \mathbb{N} \cup \{0\}, k \in \mathcal{K}(j)\}$ there exist a point $y_k^j \in Y$ and an open set Q_k^j satisfying, among many of the basic properties of dyadic cubes of $\mathbb{R}^n$, the following two which will be used in the sequel:

(a) there exist constants $a > 0$, $c > 0$ and $0 < \delta < 1$ such that $B_d(y_j^k, a\delta^j) \subseteq Q_k^j \subseteq B_d(y_j^k, c\delta^j)$ for every $(j,k) \in \mathcal{A}$;

(b) every bounded open subset of Y can be written, up to a set of μ-measure zero, as a disjoint union of Christ's cubes (see [3]).

Proof of Theorem 2.3. Fix a d-ball $B = B(x_0,R)$ on Y. Let E be a Borel subset of B. Notice that since for $x \in B$ and $r = 2R$, from the doubling property for μ we have that

$$\begin{aligned} \left(\frac{\mu(E)}{\mu(B)}\right)^p &\leq \left(\frac{\mu(B(x,r))}{\mu(B)}\right)^p \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \mathcal{X}_E\, d\mu \right)^p \\ &\leq C_1 \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \mathcal{X}_E\, d\mu \right)^p . \end{aligned}$$

Hence from (2) with $f = \mathcal{X}_E$ we have

$$\begin{aligned} \nu(B)\left(\frac{\mu(E)}{\mu(B)}\right)^p &\leq C_1 \int_B \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \mathcal{X}_E\, d\mu \right)^p d\nu(x) \\ &\leq C_1 \int_Y \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \mathcal{X}_E\, d\mu \right)^p d\nu(x) \\ &\leq C_1 C \nu(E) \\ &= C_2 \nu(E). \end{aligned}$$

The inequality obtained for $E \subseteq B$ can be rewritten as

$$\frac{\mu(E)}{\mu(B)} \leq C_2^{1/p} \left(\frac{\nu(E)}{\nu(B)}\right)^{1/p} . \tag{3}$$

Taking as E the ball with the same center of B and half its radius we obtain the first claim in the statement of Theorem 2.3, i.e., ν is also a doubling measure on (Y,d).

From the doubling property for μ and the inner and outer control of the dyadic sets by the family of d-balls, we immediately conclude an inequality which is is similar to (3) with Christ's sets instead of balls. In fact, let Q be a dyadic set and let B and $\widetilde{B}$ two concentric balls of comparable radii such that $B \subseteq Q \subseteq \widetilde{B}$, hence, for each measurable set E in Q, we have the inequalities

$$\begin{aligned}\frac{\mu(E)}{\mu(Q)} &\leq \frac{\mu(E)}{\mu(B)}\\ &\leq A\,\frac{\mu(E)}{\mu(\widetilde{B})}\\ &\leq AC_2^{1/p}\left(\frac{\nu(E)}{\nu(\widetilde{B})}\right)^{1/p}\\ &\leq AC_2^{1/p}\left(\frac{\nu(E)}{\nu(Q)}\right)^{1/p}.\end{aligned}$$

The last inequality shows that $\mu(E)/\mu(Q) < A(C_2\alpha)^{1/p}$ whenever $\nu(E)/\nu(Q) < \alpha$ and E is a measurable subset of Q, for $0 < \alpha < 1$. Applying this remark to $Q - E$ instead of E we can have that

$$\frac{\mu(E)}{\mu(Q)} \leq 1 - A(C_2\alpha)^{1/p} \quad \text{implies} \quad \frac{\nu(E)}{\nu(Q)} \leq 1 - \alpha. \tag{4}$$

for $0 < \alpha < \min\{1, /C_2A^p\}$. In fact, $\frac{\nu(E)}{\nu(Q)} > 1 - \alpha$ implies $\frac{\nu(Q-E)}{\nu(Q)} < \alpha$, so that

$$\frac{\mu(Q-E)}{\mu(Q)} < A(C_2\alpha)^{1/p},$$

which implies

$$\frac{\mu(E)}{\mu(Q)} > 1 - A(C_2\alpha)^{1/p}.$$

Let us fix such an α and let us write $\beta = 1 - A(C_2\alpha)^{1/p}$. Notice that $0 < \beta < 1$. Let us assume that ν is not absolutely continuous with respect to μ. Assume then that E is a Borel set in Y such that $\mu(E) = 0$ but $\nu(E) > 0$. Since ν is regular, there exists a open set G containing E such that $\nu(G) < \nu(E)/(1-\alpha)$. Since both measures μ and ν are doubling, the boundaries of the dyadic sets are sets of μ and ν zero measures. In particular, this fact allows to write $G = \bigcup_j Q_j \cup N$, with Q_j dyadic sets and N a set of μ and ν measures equal to zero. Hence $\nu(G) = \sum_j \nu(Q_j)$. On the other hand, since $0 = \mu(E \cap Q_j)$ for every j, we have that $\frac{\mu(E\cap Q_j)}{\mu(Q_j)} \leq \beta$. So that from (4) we get that $\nu(E \cap Q_j) \leq (1-\alpha)\nu(Q_j)$. Adding in $j \in \mathbb{N}$ we get $\nu(E) \leq (1-\alpha)\nu(G)$, which contradicts the choice of G. So that $\nu \ll \mu$. Let us write $w(x)$ to denote the Radon-Nikodym derivative of ν with respect to μ. For the proof of the last statement, namely, that $w \in A_p(Y, d, \mu)$, we only have to observe that the standard proof of Theorem 2.1 only requires the current hypothesis (2) of our Theorem 2.3. □

In the last part of this section we aim to get a version for A_p suitable for the weak star convergence of measures. In this direction we shall prove the following result.

Theorem 2.4. *Let (Y,d) be a compact metric space and let (Y,d,μ) be a space of homogeneous type. Let ν be a Borel measure on Y which is positive and finite on each d-ball of Y. Let $1<p<\infty$ be given. Then $\nu = w\,d\mu$ with $w \in A_p(Y,d,\mu)$ if and only if there exists a constant $C>0$ such that the inequality*

$$\int \left(\frac{1}{\int \varphi_{x,r}(y)\,d\mu(y)} \int |f(y)|\varphi_{x,r}(y)\,d\mu(y) \right)^p d\nu(x) \leq C \int |f(y)|^p\,d\nu(y) \tag{5}$$

holds for every $f \in \mathrm{Lip}(Y)$ and for every $r>0$, where $\varphi_{x,r}(y) = \varphi\left(\frac{d(x,y)}{r}\right)$ and φ is the continuous function on $\mathbb{R}^+$ that takes the value one in $[0,1]$, vanishes in $[2,\infty)$, and is linear in the interval $[1,2]$.

In order to prove the above result, we shall need to prove the simultaneous density of the class of Lipschitz functions on the spaces $L^p(\mu)$ and $L^p(\nu)$.

Lemma 2.5. *Let (Y,d) be a compact metric space. Assume that μ and ν are two finite Borel measures on Y. Then for every $1 \leq p,q < \infty$ we have that* $\mathrm{Lip}(Y)$ *is dense in $L^p(\mu) \cap L^q(\nu)$.*

Proof. Since μ and ν are finite, they are regular (see [6] for example). Moreover, since Y is compact we have that for any Borel set E in Y and every $\varepsilon > 0$ there exist two open sets G_1 and G_2 containing E, and two compact sets K_1 and K_2 contained in E such that we have $\mu(G_1 - K_1) < \varepsilon$ and $\nu(G_2 - K_2) < \varepsilon$. Taking $G = G_1 \cap G_2$, $K = K_1 \cup K_2$ and g defined as

$$g(x) = \frac{d(x,G^c)}{d(x,G^c) + d(x,K)}$$

we have that g is a Lipschitz function, and that

$$\int |g - \mathcal{X}_E|^p d\mu \leq 2^p \mu(G-K) < 2^p \varepsilon$$

and

$$\int |g - \mathcal{X}_E|^q d\nu \leq 2^p \nu(G-K) < 2^q \varepsilon.$$

With the standard arguments we obtain the desired result. □

Proof of Theorem 2.4. Assume first that $d\nu = w d\mu$ with $w \in A_p(Y,d,\mu)$. For each $x \in Y$, $r>0$ and $f \in \mathrm{Lip}(Y)$, we have

$$\begin{aligned} \frac{1}{\int \varphi_{x,r}(y)\,d\mu(y)} \int |f(y)|\varphi_{x,r}(y)\,d\mu(y) &\leq \frac{1}{\mu(B_d(x,r))} \int_{B_d(x,2r)} |f(y)|\,d\mu(y) \\ &\leq A\,\frac{1}{\mu(B_d(x,2r))} \int_{B_d(x,2r)} |f(y)|\,d\mu(y) \\ &\leq A\,Mf(x), \end{aligned}$$

where A is the doubling constant for μ. On the other hand, from Theorem 2.2 we have that

$$\int |Mf|^p w\,d\mu \leq C \int |f|^p w\,d\mu$$

for some constant C. Thus

$$\int \left(\frac{1}{\int \varphi_{x,r}(y)\,d\mu(y)} \int |f(y)|\varphi_{x,r}(y)\,d\mu(y) \right)^p w(x)\,d\mu(x) \leq A^p C \int |f|^p w\,d\mu,$$

as desired.

In order to show that (5) implies the absolute continuity of ν with respect to μ an that $\frac{d\nu}{d\mu}$ is an $A_p(Y,d,\mu)$ weight, using Theorem 2.3, we only have to prove that (5) for every Lipschitz function implies (2) for every function f in $L^1(Y,\mu)$. It is easy to see, using again the doubling condition for μ, that (5) for Lipschitz functions implies (2) for Lipschitz functions. In fact

$$\begin{aligned}
&\int\left(\frac{1}{\mu(B_d(x,r))}\int_{B_d(x,r)}|g(y)|\,d\mu(y)\right)^p d\nu(x)\\
&\leq A^p\int\left(\frac{1}{\int\varphi_{x,r}(y)\,d\mu(y)}\int\varphi_{x,r}(y)|g(y)|\,d\mu(y)\right)^p d\nu(x)\\
&\leq A^pC\int|g|^p d\nu
\end{aligned}$$

for every $r>0$ and every function g in $\mathrm{Lip}(Y)$.

Notice that, from the monotone convergence theorem, in order to prove (2) for general f it is enough to prove it for functions belonging to $L^\infty(\mu)\cap L^\infty(\nu)$. In fact, taking

$$h_n=\begin{cases} f & \text{if } |f|\leq n,\\ n & \text{if } |f|>n,\end{cases}$$

the inequalities

$$\int\left(\frac{1}{\mu(B_d(x,r))}\int_{B_d(x,r)}|h_n(y)|\,d\mu(y)\right)^p d\nu(x)\leq\widetilde{C}\int|h_n(y)|^p\,d\nu(y)$$

for every $n\in\mathbb{N}$, imply the same inequality with f instead of h_n. Let us assume then that $f\in L^\infty(\mu)\cap L^\infty(\nu)$. Let $\{g_k\}$ be a sequence of Lipschitz functions provided by Lemma 2.5 such that $g_k\to f$ both in the $L^p(\mu)$ and $L^p(\nu)$ norms. Set, for fixed $r>0$,

$$G_k^r(x)=\frac{1}{\mu(B_d(x,r))}\int_{B_d(x,r)}|g_k|\,d\mu$$

and

$$F^r(x)=\frac{1}{\mu(B_d(x,r))}\int_{B_d(x,r)}|f|\,d\mu.$$

Notice that

$$\begin{aligned}
\int|G_k^r-F^r|^p\,d\nu &\leq \int\left(\frac{1}{\mu(B_d(x,r))}\int_{B_d(x,r)}|g_k(y)-f(y)|^p\,d\mu(y)\right)d\nu(x)\\
&\leq \|g_k-f\|^p_{L^p(\mu)}\int\frac{d\nu(x)}{\mu(B_d(x,r))}.
\end{aligned}$$

On the other hand, the last integral is finite since $\nu(Y)\infty$ and, for $r>0$ fixed, $\mu(B_d(x,r))$ as a function of x is bounded below. In fact, let $\{B_d(x_i,r/2):i=1,\dots,I\}$ be a finite covering of the compact space Y by d-balls of radius $r/2$. Hence, given $x\in Y$ there exists $i\in\{1,\dots,I\}$ such that $d(x,x_i)<r/2$, so that $B_d(x,r)\supseteq B_d(x_i,r/2)$. Thus

$$\mu(B_d(x,r))\geq\min_{i\in\{1,\dots,I\}}B_d(x_i,r/2)>0.$$

Finally, from the above remarks we get

$$\|F^r\|_{L^p(\nu)} \leq \|G_k^r\|_{L^p(\nu)}+\|G_k^r-F^r\|_{L^p(\nu)}$$

$$
\begin{aligned}
&\leq \widetilde{C}\|g_k\|_{L^p(\nu)} + \|G_k^r - F^r\|_{L^p(\nu)} \\
&\leq \widetilde{C}\|f\|_{L^p(\nu)} + \widetilde{C}\|g_k - f\|_{L^p(\nu)} + \|G_k^r - F^r\|_{L^p(\nu)}.
\end{aligned}
$$

Letting $k \to \infty$ we obtain (2) for bounded functions. □

3. Proof of Theorem 1.2

Recall that (X, d) is a compact metric space with finite Assouad metric dimension. Let $(Y, \mu) \in \mathcal{D}(A)$ and let ν be a Borel measure on Y such that $(Y, \nu) \in \mathcal{E}$. For $1 < p < \infty$ we shall write $\boldsymbol{\nu \in \mathcal{A}_p(Y, \mu)}$ if ν satisfies (5) for some constant C, for every $r > 0$ and every $f \in \mathrm{Lip}(X)$. In the sequel we shall say that such constant C is a **Muckenhoupt constant** for ν. Notice that this definition is not *a priori* the characterization of $A_p(Y, \mu)$ given in Theorem 2.4, since there f ranges on the space $\mathrm{Lip}(Y)$ and here on the space $\mathrm{Lip}(X)$. Since both spaces $\mathrm{Lip}(X)$ and $\mathrm{Lip}(Y)$ are defined with respect to the same distance d, the trace on Y of every function in $\mathrm{Lip}(X)$ belongs to $\mathrm{Lip}(Y)$. On the other hand, since X has finite metric dimension, the basic covering lemma used to generalize the Whitney extension method for Lipschitz function, holds. This fact proves that (5) holds for every $f \in \mathrm{Lip}(X)$ if and only if (5) holds for every $f \in \mathrm{Lip}(Y)$. So that $\nu \in \mathcal{A}_p(Y, \mu)$ if and only if $\nu \ll \mu$ and $w = \frac{d\nu}{d\mu} \in A_p(Y, \mu)$. For the sake of completeness let us state the extension lemma and briefly sketch the idea of the proof that in the Euclidean case can be found in [14] (see [1] for the general setting).

Lemma 3.1. *Let (X, d) be a compact metric space with finite Assouad metric dimension. Let Y be a given proper subset of X. Then there exists a linear and continuous extension operator from $\mathrm{Lip}(Y)$ to $\mathrm{Lip}(X)$.*

Set $G = X - Y$. Let $\mathcal{W} = \{B_k\}$ be a Whitney covering of balls for G, and let $\{\phi_k : k \in \mathbb{N}\}$ be an adequate partition of unity associated to $\mathcal{W}$. Given a Lipschitz function f on Y, an extension $\widetilde{f}$ of f to X is

$$
\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in Y; \\ \sum_k f(y_k)\phi_k(x) & \text{if } x \in G, \end{cases}
$$

where $\{y_k\}$ is any sequence in Y such that y_k belongs to a fixed dilation of B_k. The function $\widetilde{f}$ has the required properties.

Theorem 1.2 will be a consequence of the following quantitative more precise statement.

Theorem 3.2. *Let $1 < p < \infty$ be given. Let $\{(Y_n, \mu_n) : n \in \mathbb{N}\}$ be a given sequence in $\mathcal{D}(A)$ such that $(Y_n, \mu_n) \xrightarrow{d_{HK}} (Y, \mu)$. Let $\{\nu_n : n \in \mathbb{N}\}$ be a sequence of measures such that $\nu_n \in \mathcal{A}_p(Y_n, \mu_n)$ with Muckenhoupt constant C for every n. If $\nu_n \xrightarrow{w*} \nu$, then $\nu \in \mathcal{A}_p(Y, \mu)$ with the same Muckenhoupt constant C.*

Hence Theorem 1.2 is a consequence of Theorems 3.2 and 2.4. Notice also that when $Y_n = Y$ and $\mu_n = \mu$ for every $n \in \mathbb{N}$, the result of Theorem 3.2 is the completeness of the class $\mathcal{A}_p(Y, \mu; C)$ of those $\mathcal{A}_p(Y, \mu)$ measures with Muckenhoupt constant C. Hence any contractive mapping on $\mathcal{A}_p(Y, \mu; C)$ with respect to the distance d_K, has a fixed point in $\mathcal{A}_p(Y, \mu; C)$. Let us also observe that without the hypothesis $(Y_n, \mu_n) \in \mathcal{E}$ contained in the definition of $\mathcal{A}_p(Y_n, \mu_n)$ the result does not hold. In fact, it is enough to take $X = [0, 1]$, d the usual distance, $d\mu = dx$ the Lebesgue measure, $Y_n = X$, $d\mu_n = d\mu = dx$ for each n, and $d\nu_n = \frac{dx}{n}$.

Proof of Theorem 3.2. Given $f \in \operatorname{Lip}(X)$, $r > 0$, $x \in X$ and $n \in \mathbb{N} \cup \{\infty\}$, let us write $\mathscr{M}_n f(x,r)$ to denote the smooth mean value

$$\mathscr{M}_n f(x,r) = \frac{1}{\int \varphi_{x,r}(y)\, d\mu_n(y)} \int |f(y)| \varphi_{x,r}(y)\, d\mu_n(y),$$

when $\int \varphi_{x,r}(y)\, d\mu_n(y) > 0$. If $\int \varphi_{x,r}(y)\, d\mu_n(y) = 0$ we define $\mathscr{M}_n f(x,r) = 0$. Here we are using the notation μ_∞ for μ. We have to prove that

$$\int (\mathscr{M}_\infty f(x,r))^p \, d\nu(x) \leq C \int |f(y)|^p \, d\nu(y), \tag{6}$$

where C is a uniform Muckenhoupt constant for the whole sequence $\{\nu_n : n \in \mathbb{N}\}$.

In order to prove (6), it is enough to show that for every $\varepsilon > 0$ there exists $N = N(\varepsilon, r, f) \in \mathbb{N}$ such that for every $n \geq N$ the inequality

$$\int (\mathscr{M}_\infty f(x,r))^p \, d\nu_n(x) \leq \varepsilon + C \int |f(y)|^p \, d\nu_n(y) \tag{7}$$

holds. In fact, once (7) is proved, since $|f|^p$ is continuous on X, the weak convergence of ν_n to ν shows that the right hand side tends to $\varepsilon + C \int |f|^p \, d\nu$ as $n \to \infty$. On the other hand, even when $(\mathscr{M}_\infty f(x,r))^p$ could be discontinuous on X, it is certainly continuous on $[Y]_{r/4}$, the $r/4$-enlargement of Y, since in this region we have that $\mathscr{M}_\infty f(x,r)$ is the quotient of the two continuous functions $\int |f| \varphi_{x,r}\, d\mu$ and $\int \varphi_{x,r}\, d\mu$. Notice that the last one is positive because φ is one on some small ball B centered at a point of Y, and since μ is a doubling measure we have $0 < \mu(B) \leq \int \varphi_{x,r}\, d\mu$. Also the Hausdorff convergence of Y_n to Y implies that for n large enough $Y_n \subseteq [Y]_{r/8}$. Hence after a continuous extension to the whole space X of the restriction of $\mathscr{M}_\infty f(x,r)$ to the closure of $[Y]_{r/8}$, we can also take limit as $n \to \infty$ to the left hand side of (7) and use again the weak star convergence of ν_n to ν in order to obtain (6), except for an arbitrarily small ε added to its right hand side.

Let us proceed to prove (7). Notice that to achieve this goal it is enough to show that for each $r > 0$ and each $f \in \operatorname{Lip}(X)$, the sequence $\mathscr{M}_n f(x,r)$ converges uniformly to $\mathscr{M}_\infty f(x,r)$ on $[Y]_{r/4}$. In fact, we have that for each $\varepsilon > 0$ there exists N which could depend on f, r and ε but not on $x \in [Y]_{r/4}$, such that

$$\begin{aligned}
\int (\mathscr{M}_\infty f(x,r))^p \, d\nu_n(x) &\leq \int |(\mathscr{M}_\infty f(x,r))^p - (\mathscr{M}_n f(x,r))^p| \, d\nu_n(x) \\
&\quad + \int (\mathscr{M}_n f(x,r))^p \, d\nu_n(x) \\
&\leq \varepsilon + \int (\mathscr{M}_n f(x,r))^p \, d\nu_n(x),
\end{aligned}$$

for every $n \geq N$. Now, since $\nu_n \in \mathcal{A}_p(Y_n, \mu_n; C)$, we have (7).

In order to prove the uniform convergence of $\mathscr{M}_n f(\cdot, r)$ to $\mathscr{M}_\infty f(\cdot, r)$ on $[Y]_{r/4}$, notice that since $\int \varphi_{x,r}\, d\mu > 0$ on the closure of $[Y]_{r/4}$, we only have to prove the uniform convergence of $\int |f(y)| \varphi_{x,r}(y)\, d\mu_n(y)$ to $\int |f(y)| \varphi_{x,r}(y)\, d\mu_\infty(y)$ on $[Y]_{r/4}$ for every $f \in \operatorname{Lip}(X)$. In fact, since $\int \varphi_{x,r}\, d\mu$ is positive and continuous on the compact $\overline{[Y]_{r/4}}$, then it has a positive lower bound c. So that for n large enough we have that $Y_n \subseteq [Y]_{r/4}$ and that $\int \varphi_{x,r}\, d\mu_n > c/2$, hence

$$\begin{aligned}
&|\mathscr{M}_n f(x,r) - \mathscr{M}_\infty f(x,r)| \\
&\leq \frac{1}{\int \varphi_{x,r}\, d\mu_n} \left| \int |f|\varphi_{x,r}\, d\mu_n - \int |f|\varphi_{x,r}\, d\mu \right| \\
&+ \frac{\int |f|\varphi_{x,r}\, d\mu}{\left(\int \varphi_{x,r}\, d\mu_n\right)\left(\int \varphi_{x,r}\, d\mu\right)} \left| \int \varphi_{x,r}\, d\mu_n - \int \varphi_{x,r}\, d\mu \right| \\
&\leq \frac{2}{c} \left| \int |f|\varphi_{x,r}\, d\mu_n - \int |f(y)|\varphi_{x,r}\, d\mu \right| + \frac{2\int |f|\, d\mu}{c^2} \left| \int \varphi_{x,r}\, d\mu_n - \int \varphi_{x,r}\, d\mu \right|,
\end{aligned}$$

which tends to zero as $n \to \infty$ uniformly on $[Y]_{r/4}$.

Let us finally to prove the uniform convergence of $\int |f|\varphi_{x,r}\, d\mu_n$ to $\int |f|\varphi_{x,r}\, d\mu_\infty$ on $[Y]_{r/4}$ for every $f \in \mathrm{Lip}(X)$. Precisely this is the main point of the use Lipschitz functions in order to test the Muckenhoupt condition. If $f \in \mathrm{Lip}(X)$, there exists $\Lambda > 0$ such that $f \in \mathrm{Lip}_\Lambda(X)$. On the other hand, for x and r fixed $\varphi_{x,r} \in \mathrm{Lip}_{1/\mathrm{r}}(X)$. Hence $g_{x,r}(y) = |f(y)|\varphi_{x,r}(y)$ belongs to $\mathrm{Lip}_{\frac{1}{\mathrm{r}}\|\mathrm{f}\|_\infty + \Lambda}(X)$, where $\|f\|_\infty = \sup_{y\in X} |f(y)|$. From the very definition of d_K and the weak star convergence of μ_n to μ we have

$$\sup_{x\in X} \left| \int g_{x,r}(y)\, d\mu_n(y) - \int g_{x,r}(y)\, d\mu_\infty(y) \right| \leq \left(\frac{1}{r}\|f\|_\infty + \Lambda \right) d_K(\mu_n, \mu_\infty) \xrightarrow[n\to\infty]{} 0.$$

In other words, $\int |f|\varphi_{x,r}\, d\mu_n$ converges uniformly to $\int |f|\varphi_{x,r}\, d\mu_\infty$ on $[Y]_{r/4}$. □

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