MULTIRESOLUTION APPROXIMATIONS AND UNCONDITIONAL BASES ON WEIGHTED LEBESGUE SPACES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. Starting from a slight modification of the dyadic sets introduced by M. Christ in [M. Christ, A T(b) theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990) 601–628] on a space of homogeneous type (X, d, μ), an MRA type structure and a Haar system $\mathcal{H}$ controled by the quasi distance d, can be constructed in this general setting in such a way that $\mathcal{H}$ is an orthonormal basis for $L^2(d\mu)$. This paper is devoted to explore under which conditions on the measure ν, the system $\mathcal{H}$ is also an unconditional basis for the Lebesgue spaces $L^p(d\nu)$. As a consequence, we obtain a characterization of these spaces in terms of the $\mathcal{H}$–coefficients.

1. Introduction

A general approach to signals and images defined on domains including the classical continuous time ($\mathbb{R}$), the n–dimensional space ($\mathbb{R}^n$), the discrete time ($\mathbb{Z}$), the discrete space ($\mathbb{Z}^n$), the sphere S^2, some fractals, etc., can be given by considering real functions defined on quasi-metric measure spaces. A considerable amount of classical analysis can be extended to quasi–metric measure spaces satisfying a doubling property, usually called spaces of homogeneous type, and even to more general settings.

The advantage of working in such a general framework is given by the wide scope of situations that the general structure of space of homogeneous type can model.

The basic disadvantage of the general setting, related to our current problem of constructing a multiresolution structure, is the lack of the classical notion of self–similarity. The main tool to recover, in a generalized fashion, the idea of self–similarity is provided by the dyadic families introduced by M. Christ in [6]. Let us mention that Haar type wavelets associated to nested partitions in an given abstract measure space are provided by M. Girardi and W. Swelden in [12]. In [5] an attempt is done to provide such a Haar type basis in spaces of homogeneous type, with a metric control of the size of the dyadic pieces. There, only an outer control is provided and, as an analytical consequence, the dyadic maximal function could be "far away" from the standard Hardy–Littlewood maximal function over balls. After being acquainted by C. Kenig about the existence of such an outstanding construction as is the one provided by M. Christ, the first author in [2] builds a

Date: March 10, 2006.

2000 *Mathematics Subject Classification.* 42C15, 42B20, 28C15.

Key words and phrases. wavelets, weighted Lebesgue spaces, A^p weights, spaces of homogeneous type.

The authors were supported in part by CAI+D(UNL), Conicet, ANPCyT and MEC grants MTM2005-08350-C03-02.

Haar type basis starting from these families of dyadic sets. In [3] we provide the comparison of the Hardy–Littlewood maximal function and the dyadic maximal function associated to these dyadic sets.

This paper is devoted to explore under which conditions on a measure ν, defined on the Borel sets of a space of homogeneous type (X, d, μ), the Haar system built on Christ's dyadic sets, is an unconditional basis for the Lebesgue space $L^p(d\nu)$ and we also give a characterization of these spaces in terms of the Haar coefficients.

As far as we know the previous work in this direction, for the Haar systems or for smooth wavelets, in the interval, in the line and in the n–dimensional euclidean space is contained in [15], [14], [9], [10], [16], [11] and [4]. In [15] A. Krantzberg and in [14] K. Kazarian deal with different aspects of the problem posed on the interval. The unbounded case is considered by J. García Cuerva and K. Kazarian in [9] and [10], P. Lemarié in [16], J. García Cuerva and J. Martell in [11], and the first two authors and F. Martín Reyes in [4]. Actually our proofs of the main results shall follow closely the general lines of those in [4].

The paper is organized as follows. In Section 2 we introduce the general geometric framework, including a slight modification to Christ's construction in order to obtain exact coverings at each level and some new properties of these families. Section 3 is devoted to introduce some basic tools of dyadic analysis including the Calderón–Zygmund decomposition and a generalization of Fefferman–Stein inequality on the boundedness of the L^p norm of the dyadic maximal function by the L^p norm of the sharp dyadic maximal function. In Section 4 we prove weighted boundedness of the dyadic maximal function and a weighted version of Fefferman–Stein type inequality which shall be used in the proof of the main results. In Section 5 we introduce the MRA type structure and the associated Haar basis. Section 6 is devoted to obtain the basic estimates for the projection operators on Lebesgue spaces. The aim of Section 7 is twofold: on one hand, and for the sake of completeness, we prove that the Haar system introduced in §5 is an unconditional basis for the L^p–Lebesgue spaces when $1 < p < \infty$. On the other hand, and as an essential tool for the proof of our main results, we obtain a dyadic version of the estimate of the sharp maximal function of a singular integral type operator in terms of the s–maximal function, see Theorem 7.2 (iii). In Sections 8 and 9 we state and prove our results concerning the weighted inequalities for the projection operators and we explore the unconditionality of the Haar basis on weighted L^p spaces $(1 < p < \infty)$.

Along this paper we shall systematically use the following notation for the Lebesgue spaces: if the underlying space of homogeneous type is (X, d, μ), we shall write L^p to denote the space $L^p(X, d\mu) = \{f : \int_X |f|^p d\mu < \infty\}$ and $\|f\|_p$ to denote the corresponding norm. When using a measure ν instead of μ, we shall explicitly write $L^p(d\nu)$ and $\|f\|_{p,d\nu}$ to denote the space and the norm associated to ν. Finally, when ν is absolutely continuous with respect to μ with density w we simply write $L^p(w)$ and $\|f\|_{p,w}$.

2. The general setting

Let X be a set. A quasi-distance on X is a nonnegative symmetric function defined on $X \times X$ such that $d(x, y) = 0$ if and only if $x = y$ and there exists a constant K such that the inequality

$$d(x, y) \leq K[d(x, z) + d(z, y)],$$

holds for every $x, y, z \in X$. A well known result due to Macías and Segovia (see [17]) provides a distance ρ and a real number α, generally larger than one, such that d is equivalent to $\rho^\alpha =: d'$.

Since a quasi-distance d on X induces a topology through the neighborhood system $\{B(x,r) : r > 0\}$ of each point $x \in X$ (see [7]), we consider on X this topology. A basic corollary of the above mentioned theorem of Macías and Segovia is the fact that for any quasi-distance d on X it is always possible to construct an equivalent quasi-distance d' such that every d'-ball is an open set.

Let (X, d) be a quasi-metric space such that the d-balls are open sets. Let μ be a Borel measure on X satisfying the doubling condition

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty \tag{2.1}$$

for some constant A, every $x \in X$ and every $r > 0$. Along this paper we shall say that (X, d, μ) is a space of homogeneous type if (X, d) is a quasi-metric space such that d-balls are open sets, and μ is a regular measure defined on a σ-algebra Σ containing the d-balls that satisfies (2.1). We will refer to the triangle constant K and the doubling constant A as the *geometric constants* of the space. Let us observe that since we are assuming that μ is a regular measure, then the space of continuous functions with bounded support is dense in L^1, so that the Lebesgue differentiation theorem holds.

Dyadic type families of subsets in metric or quasi-metric spaces have been constructed by several authors. When only the covering and nesting properties of dyadic cubes are relevant for the analytical underlying problems, exact coverings are obtained and applied by Sawyer and Wheeden [18] (see also [5]). Nevertheless this construction process as a disjunction by substraction of balls produces dyadic type sets that become very eccentric in the sense that no metric control remains in the process. In other words is impossible to get a positive constant $c < 1$ such that for every dyadic type set E there is a ball B with $cB \subset E \subset B$, where cB is the d-ball concentric with B and radius c times that of B.

With a different technique M. Christ [6] constructs a tiling sequence of the space which satisfies all the relevant properties of the usual dyadic cubes in $\mathbb{R}^n$, including the metric control of each set. The construction of M. Christ is given on a space of homogeneous type and, actually, the doubling property of the measure allows him to prove that at each level the dyadic sets provide a covering of the whole space except for a set of null measure.

Since, when dealing with *a priori* non absolutely continuous measures, we shall actually need dyadic families satisfying both, the exact covering property of Sawyer and Wheeden and the metric control of Christ, we start by proving that we can use Christ's construction followed by a disjunction process to produce a family with the desired properties.

Let (X, d, μ) be a space of homogeneous type. Take $0 < \delta < 1$ and $j \in \mathbb{Z}$. We shall say that $\mathcal{N}_j$ is a δ^j-net in X if $\mathcal{N}_j$ is a maximal δ^j-disperse subset of X. We can write $\mathcal{N}_j = \{x_k^j : k \in \mathcal{K}(j)\}$, where $\mathcal{K}(j)$ is an initial interval of natural numbers that may coincide with all of $\mathbb{N}$. Actually $\mathcal{K}(j)$ is finite for some j if and only if (X, d) is bounded. Set $\mathcal{A} = \cup_{j\in\mathbb{Z}}(\{j\} \times \mathcal{K}(j))$.

Theorem 2.1. *Let (X, d, μ) be a space of homogeneous type. Then there exist constants $a > 0$, $C > 0$, $0 < \delta < 1$, $N \in \mathbb{N}$ and a family $\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$, with $\mathcal{D}_j = \{Q_k^j : k \in \mathcal{K}(j)\}$ of Borel subsets of X satisfying the following properties*

(d.1) *for each* $j \in \mathbb{Z}$, *the sets* Q_k^j *are pairwise disjoints and* $X = \bigcup_{k\in\mathcal{K}(j)} Q_k^j$;
(d.2) $B(x_k^j, a\delta^j) \subset Q_k^j$, *for every* $(j,k) \in \mathcal{A}$;
(d.3) $Q_k^j \subset B(x_k^j, C\delta^j)$, *for every* $(j,k) \in \mathcal{A}$;
(d.4) *for every* $(j,k) \in \mathcal{A}$ *and every* $i < j$ *there exists a unique* $\ell \in \mathcal{K}(i)$ *such that* $Q_k^j \subseteq Q_\ell^i$;
(d.5) *for* $j \geq i$ *then either* $Q_k^j \subseteq Q_\ell^i$ *or* $Q_k^j \cap Q_\ell^i = \emptyset$, $k \in \mathcal{K}(j)$ *and* $\ell \in \mathcal{K}(i)$;
(d.6) *for every* $\ell \in \mathcal{K}(j-1)$ *and every* $j \in \mathbb{Z}$, $\#\{k \in \mathcal{K}(j) : Q_k^j \subset Q_\ell^{j-1}\} \leq N$;
(d.7) $\mu(\partial Q_k^j) = 0$, *for every* $(j,k) \in \mathcal{A}$, *where* ∂Q_k^j *is the boundary of* Q_k^j;
(d.8) *for each* $(j,k) \in \mathcal{A}$, *there exists a subset* $L(j,k)$ *of* $\mathcal{K}(j+1)$ *with* $1 \leq \#L(j,k) \leq N$ *such that* $Q_k^j = \bigcup_{\ell\in L(j,k)} Q_\ell^{j+1}$;
(d.9) X *is bounded if and only if there exists* $(j,k) \in \mathcal{A}$ *such that* $X = Q_k^j$;
(d.10) *there exists a constant* $\tilde{A}$ *(depending only on* K, A, C, a *and* δ*), such that for every* $(j,k) \in \mathcal{A}$, (Q_k^j, d, μ) *is a space of homogeneous type with geometric constants* K *and* $\tilde{A}$.

Proof. Let $\{\mathfrak{Q}_k^j : (j,k) \in \mathcal{A}\}$ be a Christ's dyadic family on (X, d, μ) (see [6]). This means a family of open subsets of X satisfying all properties of the theorem with the only exception that both coverings in (d.1) and (d.8) are valid except for μ-null sets. Let us start by defining the sets Q_k^0 for each $k \in \mathcal{K}(0)$. Take Q_1^0 as the closure $\overline{\mathfrak{Q}_1^0}$ of $\mathfrak{Q}_1^0$, and assuming that $2 \in \mathcal{K}(0)$ take $Q_2^0 = \overline{\mathfrak{Q}_2^0} \setminus Q_1^0$. In general, if $\ell \in \mathcal{K}(0)$ take $Q_\ell^0 = \overline{\mathfrak{Q}_\ell^0} \setminus \left(\bigcup_{i=1}^{\ell-1} Q_i^0\right)$. In this way we build the family $\mathcal{D}_0 = \{Q_k^0 : k \in \mathcal{K}(0)\}$ which obviously satisfies (d.1) with $j = 0$. To define the family $\mathcal{D}_1$ take first a fixed $Q_k^0 \in \mathcal{D}_0$ and consider the set $L(0,k)$ of those $\ell \in \mathcal{K}(1)$ such that $\mathfrak{Q}_\ell^1 \subseteq \mathfrak{Q}_k^0$. Now using the above argument in the quasi-metric space (Q_k^0, d) we obtain by closure in (Q_k^0, d) and disjunction a family $\{Q_\ell^1 : \ell \in L(0,k)\}$ satisfying (d.8) for $j = 0$. The family $\mathcal{D}_1$ is then given by $\bigcup_{k\in\mathcal{K}(0)}\{Q_\ell^1 : \ell \in L(0,k)\}$. Notice that, clearly (d.1) holds for $j = 1$. Repeating this procedure we obtain a family $\mathcal{D}_j = \{Q_k^j : k \in \mathcal{K}(j)\}$ for every $j \geq 0$ satisfying (d.1) and (d.8). For $j < 0$ and $k \in \mathcal{K}(j)$, let us define

$$Q_k^j = \bigcup_{\{i:\mathfrak{Q}_i^0 \subseteq \mathfrak{Q}_k^j\}} Q_i^0.$$

Notice that we also have

$$Q_k^j = \bigcup_{\{i:\mathfrak{Q}_i^{j+1} \subseteq \mathfrak{Q}_k^j\}} Q_i^{j+1}.$$

It is clear that (d.1) and (d.8) remains valid for $j < 0$. Let us finally observe that (d.2) to (d.7), (d.9) and (d.10) follow easily from the properties of the family $\{\mathfrak{Q}_k^j : (j,k) \in \mathcal{A}\}$ of M. Christ ([6] see also [3]). □

The family $\mathcal{D}$ constructed above share with the dyadic cubes of $\mathbb{R}^n$ even more geometric properties than those stated in Theorem 2.1. A usefull tool in the proof of the results in §5 is the following concept. Let Q be a fixed dyadic set in $\mathcal{D}$. The set

$$\mathcal{C}(Q) = \bigcup_{\{Q'\in\mathcal{D}:Q'\supseteq Q\}} Q'$$

shall be called the quadrant of X containing Q.

Lemma 2.2. *The family of quadrants satisfies the following properties*

i) *for each quadrant $\mathcal{C}$ we have that $(\mathcal{C}, d, \mu)$ is a space of homogeneous type;*
ii) *two intersecting quadrants coincide;*
iii) *there exists a purely geometric constant M such that $X = \bigcup_{i=1}^{M} \mathcal{C}_i$, with $\mathcal{C}_i$ quadrants of X;*
iv) *if $\mu(X) < \infty$ then there exists only one quadrant that coincides with a $Q \in \mathcal{D}$ and with X;*
v) *if $\mu(X) = \infty$ then for every quadrant $\mathcal{C}$ we also have $\mu(\mathcal{C}) = \infty$.*

Proof. Since each quadrant is an increasing union of dyadic sets property i) follows from (d.10)(see [1]). In order to prove ii), assume that $\mathcal{C}(Q_1) \cap \mathcal{C}(Q_2) \neq \emptyset$. Let us prove that $\mathcal{C}(Q_1) = \mathcal{C}(Q_2)$. In fact if $Q \supset Q_1$ and $Q' \supset Q_2$ and $Q' \cap Q \neq \emptyset$, from (d.5) we may conclude that, for example $Q' \supset Q$. Hence $\mathcal{C}(Q') = \mathcal{C}(Q)$ and since $\mathcal{C}(Q_1) = \mathcal{C}(Q)$ and $\mathcal{C}(Q_2) = \mathcal{C}(Q')$ we get the desired result. Notice that in order to prove that the number of quadrants is finite it will be enough to show that for some geometric constant M the family $\mathcal{F}_{\mathcal{C}}(x_o, R) = \{\mathcal{C} : \mathcal{C} \cap B(x_o, R) \neq \emptyset, \mathcal{C} \text{ quadrant}\}$ has at most M elements for every choice of $x_o \in X$ and $R > 0$. Take $j \in \mathbb{Z}$ such that $\delta^{j+1} \leq R < \delta^j$. Let us consider the family $\mathcal{F}_Q^j(x_o, R)$ of all dyadic sets $Q_k^j \in \mathcal{D}_j$ such that $Q_k^j \cap B(x_o, R) \neq \emptyset$. Since the points x_k^j corresponding to these $Q_k^j \in \mathcal{F}_Q^j(x_o, R)$ belong to a fixed dilation of $B(x_o, R)$ and since the net $\mathcal{N}_j$ is δ^j-disperse, we necessarily have a uniform bound by a geometric constant of the number of elements of the family $\mathcal{F}_Q^j(x_o, R)$. Since obviously $\#(\mathcal{F}_{\mathcal{C}}(x_o, R)) \leq \#(\mathcal{F}_Q^j(x_o, R))$ the desired result follows from the above argument. Set $M = \#\{\mathcal{C} : \mathcal{C} \text{ quadrant of } X\}$ and denote $\mathcal{C}_i, i = 1, 2, \ldots, M$ each quadrant of X. In order to finish the proof of iii) we have to prove that $X = \bigcup_{i=1}^{M} \mathcal{C}_i$. Notice that every $x \in X$ belongs to some Q_k^0, which in turn is contained in $\mathcal{C}(Q_k^0)$ so that, we have that $x \in \mathcal{C}_i$ for some $i = 1, \ldots, M$. To prove iv) use (d.9) and the well known fact that $\mu(X) < \infty$ is equivalent to the boundedness of X. Let us now prove v). Let $\mathcal{C} = \mathcal{C}(Q)$ be a given quadrant of X with $Q = Q_{k_0}^{j_0}$. Notice that from the definition of $\mathcal{C}(Q)$, there exists a function $k : \{j \leq j_0\} \to \mathbb{N}$ such that $k(j) \in \mathcal{K}(j)$ and $\mathcal{C}(Q) = \bigcup_{j \leq j_0} Q_{k(j)}^j$. We shall show that

$$B(x_{k_0}^{j_0}, \delta^j) \subset B(x_{k(j)}^j, K(1+C)\delta^j) \quad \text{for every } j \leq j_0, \tag{2.2}$$

where C is the constant in (d.3). Assume that (2.2) is true, then

$$\infty = \mu(X) = \lim_{j \to -\infty} \mu(B(x_{k_0}^{j_0}, \delta^j)).$$

Since, with a as in (d.2) we have that

$$\begin{aligned}
\mu(B(x_{k_0}^{j_0}, \delta^j)) &\leq \mu(B(x_{k(j)}^j, K(1+C)\delta^j)) \\
&\leq \tilde{A}\mu(B(x_{k(j)}^j, a\delta^j)) \\
&\leq \tilde{A}\mu(Q_{k(j)}^j),
\end{aligned}$$

and since also $\{Q_{k(j)}^j : j \leq j_0\}$ is non decreasing we see that

$$\mu(\mathcal{C}) = \lim_{j \to -\infty} \mu(Q_{k(j)}^j) = \infty,$$

as desired. Let us finally prove (2.2). Since $x_{k_0}^{j_0} \in Q_{k_0}^{j_0} \subset Q_{k(j)}^j$, $j \leq j_0$; using (d.3) we have that $d(x_{k_0}^{j_0}, x_{k(j)}^j) < C\delta^j$. Hence (2.2) follows from the triangle inequality. $\square$

The next result shall be used in the following section in order to prove the density in Lebesgue spaces of the simple functions defined in terms of these dyadic sets.

Lemma 2.3. *For every bounded open set G of X there exists a disjoint subfamily $\mathcal{G}$ of $\mathcal{D}$ such that $G = \bigcup_{\{Q \in \mathcal{G}\}} Q$.*

Proof. Let us define the family $\mathcal{G}$. Take $x \in G$ and $r > 0$ such that $B(x,r) \subset G$. Let us now pick $j \in \mathbb{Z}$ large enough so that $C\delta^j < \frac{r}{2K}$, with C the constant in (d.3). Then there exists a unique $k \in \mathcal{K}(j)$ such that $x \in Q_k^j$. Moreover $Q_k^j \subset B(x,r) \subset G$ from our choice of j. The family $\mathcal{D}_x(G) = \{Q \in \mathcal{D} : x \in Q \subset G\}$ is nonempty, since $Q_k^j \in \mathcal{D}_x(G)$, and, since G is bounded, $\mathcal{D}_x(G)$ is bounded above with the inclusion order. Now, for each $x \in G$ let us set $Q(x)$ to denote a maximal element of $\mathcal{D}_x(G)$. Let $\mathcal{G} = \{Q(x) : x \in G\}$. Then the lemma follows because of the fact that the elements of $\mathcal{G}$ are pairwise disjoint, and since $x \in Q(x)$ we have that $G = \cup_{Q \in \mathcal{G}} Q$. □

3. Basic tools from dyadic analysis

For a locally integrable function f we define the dyadic maximal function by

$$M^{dy} f(x) = \sup_{x \in Q \in \mathcal{D}} \frac{1}{\mu(Q)} \int_Q |f(y)| \, d\mu(y).$$

Notice that the operator M^{dy} defined above is the same as the one defined in [3] except for a μ-null set. Consequently, M^{dy} is of weak type (1,1) and bounded in L^p for $1 < p \leq \infty$.
Also in [3] we stated and proved a dyadic version of Calderón-Zygmund decomposition by using the open dyadic sets defined by Christ. Of course the same arguments can be applied to get a Calderón-Zygmund decomposition associated to the dyadic sets defined in Theorem 2.1. With the standard notation for mean values: $m_Q(f) = \frac{1}{\mu(Q)} \int_Q f \, d\mu$, for $Q \in \mathcal{D}$ and $m_X(f) = \frac{1}{\mu(X)} \int_X f \, d\mu$ if $\mu(X) < \infty$ and $m_X(f) = 0$ if $\mu(X) = \infty$, we have the following result.

Theorem 3.1. *Let (X, d, μ) be a space of homogeneous type. Let $f \geq 0$ be a μ-integrable function defined on X and λ a positive number with $\lambda \geq m_X(f)$. Then there exists a family $\mathcal{F} \subset \mathcal{D}$ such that*

(CZ.1) *if Q and Q' are distinct elements of $\mathcal{F}$, then $Q \cap Q' = \emptyset$;*
(CZ.2) *$m_Q(f) > \lambda$ for every $Q \in \mathcal{F}$;*
(CZ.3) *$m_{\tilde{Q}}(f) \leq \lambda$ for every $\tilde{Q} \in \mathcal{D}$ such that $Q \subsetneq \tilde{Q}$ for some $Q \in \mathcal{F}$;*
(CZ.4) *$m_{Q'}(f) \leq \lambda$ for every $Q' \in \mathcal{D}$ such that $Q' \cap (\bigcup_{Q \in \mathcal{F}} Q) = \emptyset$;*
(CZ.5) *$\{x \in X : M^{dy} f(x) > \lambda\} = \bigcup_{Q \in \mathcal{F}} Q =: \mathcal{O}_\lambda$;*
(CZ.6) *$f = g + b$, with $b = \sum_{Q \in \mathcal{F}} b_Q$ and $b_Q = [f - m_Q(f)]\chi_Q$;*
(CZ.7) *$|g(x)| \leq C\lambda$;*
(CZ.8) *$\int_X b_Q \, d\mu = 0$;*
(CZ.9) *$\|b\|_1 \leq 2\|f\|_1$.*

Another maximal operator that we shall use in what follows is the sharp dyadic maximal operator defined by

$$\begin{aligned}M^{\#,dy}f(x) &= \sup_{x\in Q\in\mathcal{D}} \frac{1}{\mu(Q)}\int_Q |f(y)-m_Q(f)|\,d\mu(y)\\ &\equiv \sup_{x\in Q\in\mathcal{D}} \inf_{a\in\mathbb{R}} \frac{1}{\mu(Q)}\int_Q |f(y)-a|\,d\mu(y).\end{aligned}$$

It is clear that $M^{\#,dy}f(x) \leq 2M^{dy}f(x)$. Even when the opposite inequality does not hold pointwise, a celebrated inequality proved by Fefferman and Stein in [8], shows in the euclidean setting that $\int_{\mathbb{R}^n}(Mf)^p \leq \int_{\mathbb{R}^n}(M^{\#}f)^p$. Since our context includes the case of spaces of finite measure, this inequality can not hold true in general, since in this case the Fefferman-Stein sharp maximal of constant functions is zero and of course the Hardy-Littlewood maximal function is not. Our result in the dyadic setting is contained in the next theorem. Let us point out that, using a covering lemma, a similar result holds for the general, non-dyadic, maximal functions of Hardy-Littlewood and Fefferman-Stein type (see [1]).

Theorem 3.2. *Let (X,d,μ) be a space of homogeneous type. Let $1 < p < \infty$ and $f \in L^p$ be a given function on X, we have the inequality*

$$\int_X [M^{dy}f(x)]^p d\mu(x) \leq C\left\{\mu(X)[m_X(|f|)]^p + \int_X [M^{\#,dy}f(x)]^p d\mu(x)\right\},$$

where the first term on the right hand side is zero when $\mu(X)=\infty$. The constant C depends only of p and the geometric constants of the space.

Proof. Let $\mathcal{F}$ be the family of disjoint dyadic sets for which $\mathcal{O}_\lambda = \{x \in X : M^{dy}f(x) > \lambda\} = \bigcup_{Q\in\mathcal{F}} Q$ given by (CZ.5). Let us start by proving that there exists a purely geometric constant $C > 0$ such that the inequality

$$\mu(\{x\in Q : M^{dy}f(x) > 2\lambda, M^{\#,dy}f(x) < \gamma\lambda\}) \leq C\gamma\mu(Q), \tag{3.1}$$

holds for every $Q \in \mathcal{F}$ and every $\gamma > 0$. Since we are dealing with dyadic sets only, we can follow *mutatis mutandi* the dyadic euclidean case (see [13], for example). In fact, since each set $Q \in \mathcal{F}$ is maximal with the property $m_Q(|f|) > \lambda$ given by (CZ.2), we have that for any $Q' \in \mathcal{D}$ with $Q' \supsetneq Q$, $m_{Q'}(|f|) \leq \lambda$. Hence for $x \in Q$ such that $M^{dy}f(x) > 2\lambda$, we have

$$\begin{aligned}2\lambda &< \sup_{\substack{x\in Q^*\subseteq Q\\ Q^*\in\mathcal{D}}} \frac{1}{\mu(Q^*)}\int_{Q^*}|f|d\mu\\ &= \sup_{x\in Q^*\in\mathcal{D}} \frac{1}{\mu(Q^*)}\int_{Q^*}|\chi_Q f|d\mu\\ &= M^{dy}(\chi_Q f)(x).\end{aligned}$$

Let us denote by $\tilde{Q}$ the first ancestor of Q. If $x \in Q$ with $M^{dy}f(x) > 2\lambda$, since $m_{\tilde{Q}}(|f|) \leq \lambda$, $M^{dy}\chi_Q(x) \leq 1$, and $M^{dy}(\chi_Q f)(x) > 2\lambda$, we necessarily have $M^{dy}[(f - m_{\tilde{Q}}(f))\chi_Q](x) > \lambda$. Notice that the set on the left hand side of (3.1) has μ-measure zero if $\operatorname{ess\,inf}_Q M^{\#,dy}f \geq \gamma\lambda$, so that the desired inequality holds trivially. Assume then that $\operatorname{ess\,inf}_Q M^{\#,dy}f < \gamma\lambda$. Since M^{dy} is of weak type (1.1) with constant equal to one, we have that

$$\begin{aligned}
\mu(\{x \in Q : M^{dy} f(x) > 2\lambda, M^{\#,dy} f(x) < \gamma\lambda\})& \\
&\leq \mu(\{x \in Q : M^{dy}[(f - m_{\tilde{Q}}(f))\chi_Q](x) > \lambda\}) \\
&\leq \frac{1}{\lambda} \int_X |f - m_{\tilde{Q}}(f)|\chi_Q \, d\mu \\
&\leq \frac{C}{\lambda} \mu(Q) \frac{1}{\mu(\tilde{Q})} \int_{\tilde{Q}} |f - m_{\tilde{Q}}(f)| \, d\mu \\
&\leq \frac{C}{\lambda} \mu(Q) M^{\#,dy} f(x),
\end{aligned}$$

for every $x \in \tilde{Q} \supset Q$. Hence

$$\begin{aligned}
\mu(\{x \in Q : M^{dy} f(x) > 2\lambda,\ M^{\#,dy} f(x) < \gamma\lambda\}) &\leq \lambda^{-1} \, C \, \mu(Q) \operatorname{ess\,inf}_Q M^{\#,dy} f \\
&< C \, \mu(Q) \, \gamma.
\end{aligned}$$

By adding in both sides of (3.1) over the sets of the family $\mathcal{F}$ we obtain the inequality

$$\begin{aligned}
\mu(\{x \in X : M^{dy} f(x) > 2\lambda, M^{\#,dy} f(x) < \gamma\lambda\})& \\
&\leq C \, \gamma \, \mu(\{x \in X : M^{dy} f(x) > \lambda\})
\end{aligned} \tag{3.2}$$

for every $\gamma > 0$ in the case in which $\mathcal{F} = \mathcal{F}_\lambda \neq \emptyset$. Since in both sides of (3.2) the sets become empty if $\mathcal{F} = \emptyset$, the above inequality holds for every $\gamma > 0$ and every $\lambda > 0$. Let us estimate the L^p norm of the dyadic maximal function using (3.2),

$$\begin{aligned}
\|M^{dy} f\|_p^p &= p \int_0^\infty t^{p-1} \mu(\{M^{dy} f > t\}) dt \\
&\leq p \left(\int_0^{m_X(|f|)} t^{p-1} dt \right) \mu(X) \\
&\quad + p \int_{m_X(|f|)}^\infty t^{p-1} \mu(\{M^{dy} f > t, M^{\#,dy} f < \gamma \frac{t}{2}\}) dt \\
&\quad + p \int_{m_X(|f|)}^\infty t^{p-1} \mu(\{M^{\#,dy} f \geq \gamma \frac{t}{2}\}) dt \\
&\leq \mu(X)[m_X(|f|)]^p + p \, C \, \gamma \int_0^\infty t^{p-1} \mu(\{M^{dy} f > \frac{t}{2}\}) dt + \|M^{\#,dy} f\|_p^p \\
&= \mu(X)[m_X(|f|)]^p + p \, C \, \gamma \|M^{dy} f\|_p^p + \|M^{\#,dy} f\|_p^p.
\end{aligned}$$

Since $f \in L^p$ then $M^{dy} f \in L^p$, so that by choosing γ small enough ($\gamma = \frac{1}{2p\,C}$ will do) we obtain the desired inequality. $\square$

We would like to point out that, as in the euclidean case, less restrictive conditions on f are enough to prove the above theorem. In particular the finiteness of $\|M^{dy} f\|_{p_0}$ for some $1 \leq p_0 < p < \infty$ is sufficient. We shall actually use a weighted version of inequality (3.1) applied to a function f which satisfies the strong hypothesis in weighted form.

4. Weighted boundedness of the dyadic Hardy-Littlewood and Fefferman-Stein maximal functions

In this section we shall obtain weighted inequalities for the two dyadic maximal functions associated to the family $\mathcal{D}$ defined §2. The first one shall be used to get a weighted estimate for the projection operators (Theorems 8.1 and 8.2) and the second, given in terms of the dyadic sharp maximal function, shall be used to get weighted estimates for the singular integral operators involved in the proof of unconditionality of the Haar basis in weighted L^p spaces.

Associated to a given dyadic family we define, as usual, a class of Muckenhoupt type weight functions. A non-negative, measurable and locally integrable function w defined on the space of homogeneous type (X, d, μ), is said to be a dyadic Muckenhoupt weight of class A_p^{dy}, $1 < p < \infty$ if the inequality

$$\left(\int_Q w\, d\mu\right)\left(\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} \leq C\mu(Q)^p,$$

holds for some constant C and every dyadic set $Q \in \mathcal{D}$. We say that $w \in A_1^{dy}$ if there exists a constant C such that the inequality

$$\frac{1}{\mu(Q)}\int_Q w\, d\mu \leq C \text{ess inf}_Q w,$$

holds for every dyadic set Q. Let us also define $A_\infty^{dy} = \bigcup_{p>1} A_p^{dy}$. As we proved in [3], the basic property $A_p \Rightarrow A_{p-\epsilon}$, holds in this setting so that the A_∞^{dy} can be rephrased in the following way: $w \in A_\infty^{dy}$ if and only if there exist C and δ positive such that the inequality

$$\frac{w(E)}{w(Q)} \leq C\left(\frac{\mu(E)}{\mu(Q)}\right)^\delta$$

holds for every $Q \in \mathcal{D}$ and every measurable subset E of Q.

Theorem 4.1. *Let (X, d, μ) be a space of homogeneous type. Then*

(i) *If $w \in A_p^{dy}$, $1 \leq p < \infty$, the weak type (p,p) inequality*

$$w(\{x \in X : M^{dy}f(x) > \lambda\}) \leq \frac{C}{\lambda}\, \|f\|_{p,w}^p,$$

holds for some positive constant C, for every $\lambda > 0$ and every locally integrable function f.

(ii) *If $w \in A_p^{dy}$, $1 < p < \infty$, there exists a constant C such that the inequality*

$$\|M^{dy}f\|_{p,w} \leq C\, \|f\|_{p,w},$$

holds for every locally integrable function f.

iii) *If $w \in A_\infty^{dy}$, $1 < p < \infty$, there exists a constant C depending on p, on the A_∞^{dy} constants of w and on the geometric constants of the space such that the inequality*

$$\int_X [M^{dy}f]^p w d\mu \leq C\{w(X)[m_X|f|]^p + \int_X [M^{\#,dy}f]^p w d\mu\},$$

holds for every f such that $M^{dy}f \in L^p(w)$.

Proof. (i) If $w \in A_1^{dy}$, from (CZ.5), (CZ.1) and (CZ.2)we get

$$\begin{aligned} w(\{x \in X : M^{dy}f(x) > \lambda\}) &= \sum_{Q\in\mathcal{F}} \frac{w(Q)}{\mu(Q)}\mu(Q) \\ &\leq \frac{1}{\lambda}\sum_{Q\in\mathcal{F}} \frac{w(Q)}{\mu(Q)}\int_Q |f|\,d\mu \\ &\leq \frac{C}{\lambda}\sum_{Q\in\mathcal{F}}\int_Q |f|w\,d\mu \\ &\leq \frac{C}{\lambda}\int_X |f|w\,d\mu. \end{aligned}$$

In a similar way if $w \in A_p^{dy}$, $1 < p < \infty$, and applying Hölder inequality we get that

$$\begin{aligned} w(\{x \in X : M^{dy}f(x) > \lambda\}) &= \sum_{Q\in\mathcal{F}} \frac{w(Q)}{\mu(Q)^p}\mu(Q)^p \\ &\leq \frac{1}{\lambda^p}\sum_{Q\in\mathcal{F}} \frac{w(Q)}{\mu(Q)^p}\left(\int_Q f w^{1/p}w^{-1/p}\,d\mu\right)^p \\ &\leq \frac{C}{\lambda^p}\sum_{Q\in\mathcal{F}} \frac{w(Q)}{\mu(Q)^p}\left(\int_Q |f|^p w\,d\mu\right)\left(\int_Q w^{-\frac{1}{p-1}}\,d\mu\right)^{p-1} \\ &\leq \frac{C}{\lambda^p}\int_X |f|^p w\,d\mu. \end{aligned}$$

(ii) Using the Reverse Hölder inequality (see Lemma 5.2 in [3]) we get an $\epsilon > 0$ such that $w \in A_p^{dy} \Rightarrow w \in A_{p-\epsilon}^{dy}$. Now, from Marcinkiewicz interpolation theorem we obtain (ii) by standard arguments.

(iii) We only have to apply the A_∞^{dy} condition to inequality (3.1) in order to obtain $w(\{Q : M^{dy}f > 2\lambda, M^{\#,dy}f < \gamma\lambda\}) \leq C\gamma^\delta w(Q)$, where δ is the exponent in the A_∞^{dy} condition for w in (4). Adding for all $Q \in \mathcal{F}$ we obtain

$$w(\{M^{dy}f > 2\lambda, M^{\#,dy}f < \gamma\lambda\}) \leq C\gamma^\delta w(\{M^{dy}f > \lambda\}).$$

The desired inequality follows by estimating the distribution function of $M^{dy}f$ with respect to $wd\mu$, as we did in the proof of Theorem 3.2 using now the extra hypothesis that $M^{dy}f \in L^p(w)$. □

5. Multiresolution Analysis and Haar basis for L^2 induced by a dyadic family $\mathcal{D}$

Let $\mathcal{D}$ be a family of dyadic sets given by Theorem 2.1. For each $j \in \mathbb{Z}$ let us define V_j as the closed subspace of L^2 given by

$$V_j = \{f \in L^2 : f \text{ is } \mu \text{ a.e. constant on each } Q_k^j \in \mathcal{D}_j\}$$

The first purpose of this section is to prove the next result containing some of the basic properties that the sequence $\{V_j : j \in \mathbb{Z}\}$ shares with the standard MRA structures on Euclidean spaces.

Theorem 5.1. *The sequence $\{V_j\}$ satisfies the following MRA properties*

i) *for every $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$;*

ii) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2$;
iii) a) $\cap_{j\in\mathbb{Z}} V_j$ *is the one dimensional space of all constant functions on* X *if* $\mu(X) < \infty$,
b) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$ *if* $\mu(X) = \infty$.

For the proof of ii) we shall make use of the next result.

Lemma 5.2. *The linear span of the set* $\{\chi_Q : Q \in \mathcal{D}\}$ *is dense in each* L^p *when* $p < \infty$.

Proof. Let $f \in L^p$, $f \geq 0$, with $p < \infty$ and $\epsilon > 0$ given. Pick $x_o \in X$ and $R > 0$ such that $\|f\chi_{B^c(x_o,R)}\|_p^p < \epsilon$ and set $h = f\chi_{B(x_o,R)}$. Following the standard arguments the function h can be approximated, in an increasing way, by simple functions of bounded disjoint Borel subsets. Hence, there exists $g = \sum_{n=1}^N \beta_n \chi_{E_n}$ with E_n bounded Borel subsets of X such that $\|h - g\|_p^p < \epsilon$. From the regularity of μ and Lemma 2.3, for each $n = 1, \ldots, N$ there exists a bounded open set G_n such that $E_n \subset G_n$, $\mu(G_n \setminus E_n) < \epsilon$, and $G_n = \cup_{Q\in\mathcal{G}_n} Q$ with $\mathcal{G}_n \subset \mathcal{D}$. Assume that $\mathcal{G}_n = \{Q(n,l) : l \in L(n)\}$ where $L(n)$ is an initial interval in $\mathbb{N}$ which generally coincides with $\mathbb{N}$. Thus taking, for $J \in \mathbb{N}$

$$\psi_J(x) = \sum_{n=1}^N \beta_n \sum_{\{l\in L(n),\, l\leq J\}} \chi_{Q(n,l)}(x)$$

for J large enough, we have that $\|g - \psi_J\|_p^p < \epsilon\left(\sum_{n=1}^N \beta_n\right)^p$ and that ψ_J belongs to the linear span of $\{\chi_Q : Q \in \mathcal{D}\}$ as desired. $\square$

Proof of the Theorem 5.1. Property i) follows from (d.8). In order to prove ii) take $f \in L^2$ and $\epsilon > 0$. From Lemma 5.2 there exists $\psi = \sum_{n=1}^N \beta_n \chi_{Q(n)}$ with $Q(n) \in \mathcal{D}$ such that $\|\psi - f\|_2 < \epsilon$. Notice finally that we may regard ψ as a function of V_j with $j = \max\{i : Q(n) = Q_k^i$ for $n = 1, \ldots, N$ and some $k \in K(i)\}$. In fact, since $Q(n)$ can be decomposed as a disjoint finite union of dyadic set in $\mathcal{D}_j$, we have that $\psi(x)$ is a linear combination of indicator functions of dyadic sets in $\mathcal{D}_j$. In order to prove iii) a) applying (d.9) we get that $X = Q_1^{j_o}$, for some j_o, then we see that $V_{j_o} = \{f : f$ is constant on X $\}$. Since $V_j = V_{j_o}$ for every $j \leq j_o$ and $V_j \supset V_{j_o}$ for $j \geq j_o$, we obtain $\cap_{j\in\mathbb{Z}} V_j = V_{j_o}$. Let us finally prove iii) b). Take $f \in \cap_{j\in\mathbb{Z}} V_j$. Since $f \in V_o$ we have that f is constant on each Q_k^o. If some of these constants were different from zero, we should necessarily have that f would take the same constant value on the whole quadrant containing Q_k^o. This is impossible for an L^2 function since, from part v) of Lemma 2.2, each quadrant has infinite measure. $\square$

The second aim of this section is to introduce Haar systems based on the dyadic sets. The construction of the Haar system from the spaces $\{V_j,\ j \in \mathbb{Z}\}$ was given in [2] (see also [5] and [12]). We shall sketch this construction.
Let us consider a fixed pair $(j,k) \in \mathcal{A} = \cup_{j\in\mathbb{Z}}(\{j\} \times \mathcal{K}(j))$. By (d.8) there exists $L(j,k) \subset \mathcal{K}(j+1)$ such that

$$Q_k^j = \bigcup_{\ell\in L(j,k)} Q_\ell^{j+1}$$

with $1 \leq \#L(j,k) \leq N$. Let us assume that $\#L(j,k) > 1$ and take $L'(j,k) = L(j,k) - \{\ell_o\}$, for some $\ell_o \in L(j,k)$, for example let us take ℓ_o to be the first element in $L(j,k)$. The vector space V_{j+1}^k of all functions on Q_k^j which are constant on each Q_ℓ^{j+1}, $\ell \in L(j,k)$; has the family $\left\{\chi_{Q_k^j}\right\} \bigcup \left\{\chi_{Q_\ell^{j+1}} : \ \ell \in L'(j,k)\right\}$ as a linear basis. Applying the Gram-Schmidt orthonormalization process we get an orthonormal basis of V_{j+1}^k

$$\mathcal{B}_{j+1}^k = \left\{(\mu(Q_k^j))^{-1/2}\chi_{Q_k^j}\right\} \bigcup \left\{h_{j,k}^\ell : \ \ell \in L'(j,k)\right\}.$$

If, on the other hand, $\#L(j,k) = 1$, we have that the dimension of V_{j+1}^k is equal to one and of course that $\mathcal{B}_{j+1}^k = \{(\mu(Q_k^j))^{-1/2}\chi_{Q_k^j}\}$ is the orthonormal basis for V_{j+1}^k. For $j \in \mathbb{Z}$ we define W_j as the L^2-closure of the linear span of the set $\{h_{j,k}^\ell : \ k \in \mathcal{K}(j) \text{ with } \#L(j,k) > 1 \text{ and } \ell \in L'(j,k)\}$ and $W_j = \{0\}$ if $\#L(j,k) = 1$ for every $k \in \mathcal{K}(j)$. Clearly we get that $V_{j+1} = V_j \oplus W_j$. On the other hand we have that $\oplus_{j\in\mathbb{Z}} W_j = L^2$ if $\mu(X) = \infty$ and $\oplus_{j\in\mathbb{Z}} W_j = L_0^2 = \{f \in L^2 : \int f d\mu = 0\}$ if $\mu(X) < \infty$. For the sake notational simplicity we shall keep writing $\mathcal{L}^p$, $(p \geq 1)$ in order to denote the space L^p when $\mu(X) = \infty$ and the space $L_0^p = \{f \in L^p : \int f d\mu = 0\}$ if $\mu(X) < \infty$. Set $\tilde{\mathcal{A}} = \{(j,k) \in \mathcal{A} : \#L(j,k) > 1\}$. The family

$$\mathcal{H} = \{h = h_{j,k}^\ell : (j,k) \in \tilde{\mathcal{A}} \text{ and } \ell \in L'(j,k)\}$$

is called a *Haar system induced on (X,d,μ) by the dyadic family $\mathcal{D}$*. By construction, for each $(j,k) \in \tilde{\mathcal{A}}$ and $\ell \in L'(j,k)$ the functions $h_{j,k}^\ell$ satisfy the following properties:

(h.1) $\{x \in X : h_{j,k}^\ell(x) \neq 0\} \subseteq Q_k^j$;

(h.2) $h_{j,k}^\ell$ is constant on each $Q_\ell^{j+1} \subset Q_k^j$;

(h.3) $\int h_{j,k}^\ell \, d\mu = 0$.

Of course $\mathcal{H}$ is an orthonormal basis for $\mathcal{L}^2$, so that we have both

$$f = \sum_{h\in\mathcal{H}} \langle f,h\rangle \, h \tag{5.1}$$

in the L^2 sense for $f \in \mathcal{L}^2$ and

$$\|f\|_2^2 = \sum_{h\in\mathcal{H}} |\langle f,h\rangle|^2. \tag{5.2}$$

Notice, by the way, that in the case of finite measure we have that for any $f \in L^2$,

$$f = m_X(f) + \sum_{h\in\mathcal{H}} \langle f,h\rangle \, h \quad \text{and} \tag{5.3}$$

$$\|f\|_2^2 = \frac{\left(\int_X f d\mu\right)^2}{\mu(X)} + \sum_{h\in\mathcal{H}} |\langle f,h\rangle|^2. \tag{5.4}$$

6. The Projections as Operators on L^p

We shall denote by P_j the Hilbert projection of L^2 onto V_j. The explicit series form of this operator is given, in the L^2 sense, by

$$P_j f = \sum_{k\in\mathcal{K}(j)} \langle f, \varphi_{j,k}\rangle \, \varphi_{j,k},$$

where $\varphi_{j,k} = (\mu(Q^j_k))^{-\frac{1}{2}} \chi_{Q^j_k}$. Moreover, from (d.1) the function

$$P_j f(x) = \sum_{k \in \mathcal{K}(j)} \langle f, \varphi_{j,k} \rangle \, \varphi_{j,k}(x) \tag{6.1}$$

is well defined for every $x \in X$. Notice that for $f \in L^p$ $(1 \leq p < \infty)$ the coefficients $\langle f, \varphi_{j,k} \rangle$ of f are still well defined since each $\varphi_{j,k}$ belongs to L^∞ and has bounded support. Let us check that $P_j f(x)$ is an L^p function and, moreover, that $\|P_j f\|_p \leq \|f\|_p$. In fact,

$$\begin{aligned} \|P_j f\|_p^p &= \sum_{k \in \mathcal{K}(j)} \left(\frac{1}{\mu(Q^j_k)} \int_{Q^j_k} |f| \, d\mu \right)^p \mu(Q^j_k) \\ &\leq \sum_{k \in \mathcal{K}(j)} \int_{Q^j_k} |f|^p \, d\mu \\ &= \|f\|_p^p. \end{aligned}$$

Properties ii) and iii) a) and b) in Theorem 5.1 allows us to show that P_j is an approximate identity for $j \to +\infty$ and that P_j converges to zero or to a constant when $j \to -\infty$, both in the L^2 norm and pointwise. The next result contains the L^p and pointwise convergence for $1 \leq p < \infty$ and even when the tools are standard, we shall briefly sketch how the geometric properties of the dyadic sets allow us to use those analytic techniques.

Theorem 6.1 ($j \to +\infty$)**.** *For $1 \leq p < \infty$ we have that $P_j f \to f$ as $j \to +\infty$ in the L^p norm and almost everywhere for every $f \in L^p$.*

Theorem 6.2 ($j \to -\infty$)**.** *For every $f \in L^p$ we have that*

(a) *if $\mu(X) = \infty$ and $1 \leq p < \infty$, then $P_j f \to 0$ as $j \to -\infty$ pointwise;*
(b) *if $\mu(X) = \infty$ and $1 < p < \infty$, then $P_j f \to 0$ as $j \to -\infty$ in the L^p norm;*
(c) *if $\mu(X) < \infty$ and $1 \leq p < \infty$, then there exists $j_o \in \mathbb{Z}$ such that $P_j f = \frac{1}{\mu(X)} \int_X f d\mu$ for every $j \leq j_o$.*

Proof of Theorem 6.1($j \to +\infty$). If g is in the linear span of $\{\chi_Q : Q \in \mathcal{D}\}$, we have that, for j large enough, $P_j g = g$ μ a.e.. So that from Lemma 5.2, given $f \in L^p$ and $\epsilon > 0$, pick g in span$\{\chi_Q : Q \in \mathcal{D}\}$ with $\|f - g\|_p < \epsilon$, then

$$\begin{aligned} \|P_j f - f\|_p &\leq \|P_j (f - g)\|_p + \|P_j g - g\|_p + \|g - f\|_p \\ &\leq 2\|g - f\|_p \\ &< 2\epsilon, \end{aligned}$$

for j large enough. In order to prove the pointwise convergence of $P_j f$ to f for $f \in L^p$ and $1 \leq p < \infty$, we only have to observe that the maximal operator of the projections

$$P^* f(x) = \sup_{j \in \mathbb{Z}} |P_j f(x)| \tag{6.2}$$

is bounded above by $M^{dy} f(x)$ which is of weak type (1,1) and bounded in L^p for $1 < p \leq \infty$. $\square$

Proof of Theorem 6.2($j \to -\infty$). (a) Given $x \in X$ there exists a sequence $\{Q^j_{k(j,x)} \in \mathcal{D} : j \in \mathbb{Z}\}$ such that $x \in Q^j_{k(j,x)}$. Notice that $\mathcal{C} = \bigcup Q^j_{k(j,x)}$ is a quadrant of X, precisely is the only quadrant containing x. Since we are assuming that $\mu(X) = \infty$, from Lemma 2.2 we also have that $\mu(\mathcal{C}) = \infty$ and so $\mu(Q^j_{k(j,x)}) \to \infty$, $j \to -\infty$. Then, from Jensen inequality

$$|P_j f(x)| \leq \left[\mu(Q^j_{k(j,x)})\right]^{-\frac{1}{p}} \|f\|_p,$$

thus $P_j f(x) \to 0$ as $j \to -\infty$. (b) The L^p convergence follows from the pointwise convergence and the dominated convergence theorem, since $|P_j f(x)| \leq M^{dy} f(x) \in L^p$. (c) Follows from (d.9) $\square$

Let us remark that, since $\int P_j f d\mu = \int f d\mu$, the L^1 convergence of $P_j f$ to zero when $j \to -\infty$ does not hold true when $\mu(X) = \infty$ and $\int_X f d\mu \neq 0$.

7. The Haar system as an unconditional basis of $\mathcal{L}^p$

As in [5] (see also [12]) we can prove the following result for the space $\mathcal{L}^p$, $1 < p < \infty$.

Theorem 7.1. *Let (X, d, μ) be a space of homogeneous type, $1 < p < \infty$ and let $\mathcal{H} = \{h = h^\ell_{j,k} : (j,k) \in \tilde{\mathcal{A}}$ and $\ell \in L'(j,k)\}$ be a Haar system induced by a dyadic family $\mathcal{D}$ defined in §5. Then*

(i) *$\mathcal{H}$ is an unconditional basis for $\mathcal{L}^p$.*
(ii) *There exist two constants C_1 and C_2 such that for all $f \in \mathcal{L}^p$*

$$C_1 \|f\|_p \leq \left\| \left(\sum_{h \in \mathcal{H}} |\langle f, h\rangle|^2 \, |h|^2 \right)^{1/2} \right\|_p \leq C_2 \, \|f\|_p. \tag{7.1}$$

For the sake of completeness we shall write with some detail the proof of Theorem 7.1, let $\tilde{\mathcal{A}}^F$ be a finite subset of $\tilde{\mathcal{A}}$ and let us consider $\epsilon = \{\epsilon^\ell_{j,k}\}$ a sequence such that each $\epsilon^\ell_{j,k}$ equals $+1$ or -1. Now, we define the operators

$$T_{\tilde{\mathcal{A}}^F,\epsilon} f(x) = \sum_{(j,k) \in \tilde{\mathcal{A}}^F} \sum_{\ell \in L'(j,k)} \epsilon^\ell_{j,k} \, \langle f, h^\ell_{j,k}\rangle \, h^\ell_{j,k}(x). \tag{7.2}$$

Observe that the operators $T_{\tilde{\mathcal{A}}^F,\epsilon}$ are well defined for every locally integrable function f and that it takes values on $\mathcal{L}^p$ for $1 \leq p \leq \infty$.

Even when, strictly speaking, these operators are not Calderón-Zygmund operators because they have no smoothness, the basic Calderón-Zygmund technique can be applied to obtain the classical L^p estimates.

Theorem 7.2. *Let $T_{\tilde{\mathcal{A}}^F,\epsilon}$ be the operators defined in (7.2). Then*

(i) *there exists a constant $C_1 > 0$, independent of ϵ and $\tilde{\mathcal{A}}^F$, such that the inequality*

$$\mu(\{x \in X : |T_{\tilde{\mathcal{A}}^F,\epsilon} f(x)| > \lambda\}) \leq \frac{C_1}{\lambda} \, \|f\|_1,$$

holds for every $\lambda > 0$ and every locally integrable function f;

(ii) *for each $1 < p < \infty$ there exists C_p independent of ϵ and $\tilde{\mathcal{A}}^F$, such that the inequality*

$$\|T_{\tilde{\mathcal{A}}^F,\epsilon} f\|_p \leq C_p \, \|f\|_p,$$

holds for every locally integrable function f;

(iii) *for each $1 < s < \infty$ there exists a constant C_s, independent of ϵ and $\tilde{\mathcal{A}}^F$, such that the inequality*

$$M^{\#,dy}(T_{\tilde{\mathcal{A}}^F,\epsilon}f)(x) \le C_s[M^{dy}(|f|^s)(x)]^{1/s},$$

holds for every locally integrable function f.

Proof. First, notice that from (5.2) and (5.4) we get that the operators $T_{\tilde{\mathcal{A}}^F,\epsilon}$ are uniformly bounded on L^2, moreover $\|T_{\tilde{\mathcal{A}}^F,\epsilon}f\|_2 \le \|f\|_2$. In fact, as we observed after the definition of $T_{\tilde{\mathcal{A}}^F,\epsilon}$, for any locally integrable function f we have that $T_{\tilde{\mathcal{A}}^F,\epsilon}f \in \mathcal{L}^2$. Hence from (5.2) we have that

$$\begin{aligned}\|T_{\tilde{\mathcal{A}}^F,\epsilon}f\|_2^2 &\le \sum_{(j,k)\in\tilde{\mathcal{A}}} \sum_{\ell\in L'(j,k)} |\epsilon_{j,k}^{\ell}|^2 \, |\langle f, h_{j,k}^{\ell}\rangle| \\ &= \sum_{h\in\mathcal{H}} |\langle f, h\rangle|,\end{aligned}$$

which from (5.4) is bounded by $\|f\|_2^2$.
(i) We may assume that $f \ge 0$ and $f \in L^1$. The inequalities $0 < \lambda \le m_X(f)$ imply, on one hand, that $\mu(X) < \infty$ and, on the other hand that $\mu(\{|T_{\tilde{\mathcal{A}}^F,\epsilon}f| > \lambda\}) \le \mu(X) \le \frac{1}{\lambda}\|f\|_1$. Hence we may also assume that $\lambda > m_X(f)$, so that we are in position to obtain the Calderón-Zygmund decomposition for f given in Theorem 3.1. Then, we write $f = g + b$ and we have that

$$\mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}f(x)| > \lambda\}) \le \mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}g(x)| > \lambda/2\}) + \mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}b(x)| > \lambda/2\}).$$

Notice that $g \in L^2$, since from (CZ.7) and (CZ.9) $\int_X |g|^2\,d\mu \le C\lambda\int_X |g|d\mu \le 3C\lambda\int_X |f|d\mu$. Hence by Chebyshev's inequality and the L^2 boundedness of $T_{\tilde{\mathcal{A}}^F,\epsilon}$ we get that

$$\begin{aligned}\mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}g(x)| > \lambda/2\}) &\le \frac{4}{\lambda^2}\int_X |T_{\tilde{\mathcal{A}}^F,\epsilon}g|^2\,d\mu \\ &\le \frac{4}{\lambda^2}\int_X |g|^2\,d\mu \\ &\le \frac{12\,C}{\lambda}\int_X |f|\,d\mu.\end{aligned}$$

To estimate $\mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}b(x)| > \lambda/2\})$ we shall show that $\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}b(x)| > \lambda/2\} \subset \mathcal{O}_\lambda$ (see CZ.5).

Assume that the inclusion is true. Then by (CZ.2) and (CZ.1) we get

$$\begin{aligned}\mu(\{x : |T_{\tilde{\mathcal{A}}^F,\epsilon}b(x)| > \lambda/2\}) \le \mu(\mathcal{O}_\lambda) &\le \sum_{Q\in\mathcal{F}} \mu(Q) \\ &\le \frac{1}{\lambda}\sum_{Q\in\mathcal{F}}\int_Q f\,d\mu \le \frac{1}{\lambda}\,\|f\|_1.\end{aligned}$$

To show the inclusion it is enough to prove that $T_{\tilde{\mathcal{A}}^F,\epsilon}b(x) = 0$ if $x \notin \mathcal{O}_\lambda$. In fact, since $\sum_{Q\in\mathcal{F}} b_Q$ converges in L^1 to b and since $\tilde{\mathcal{A}}^F$ is finite, we have

$$\begin{aligned} T_{\tilde{\mathcal{A}}^F,\epsilon}b(x) &= \sum_{Q\in\mathcal{F}} T_{\tilde{\mathcal{A}}^F,\epsilon}b_Q(x) \\ &= \sum_{Q\in\mathcal{F}} \sum_{(j,k)\in\tilde{\mathcal{A}}^F} \sum_{\ell\in L'(j,k)} \epsilon^\ell_{j,k} h^\ell_{j,k}(x) \int_X b_Q(y) h^\ell_{j,k}(y)\, d\mu(y). \end{aligned}$$

Recall that $\{x \in X : h^\ell_{j,k} \neq 0\} \subseteq Q^j_k \in \mathcal{D}$ and $Q \in \mathcal{F} \subset \mathcal{D}$. If $Q \cap Q^j_k = \emptyset$ the integral is zero. If $Q^j_k \subseteq Q$, since $x \notin \mathcal{O}_\lambda$, $x \notin Q$ then $x \notin Q^j_k$ and $h^\ell_{j,k}(x) = 0$. If, $Q \subset Q^j_k$ but $Q \neq Q^j_k$, then the support of b_Q is contained in a part of Q^j_k in which $h^\ell_{j,k}$ is constant, but since $\int b_Q d\mu = 0$ we have, again, that the integral $\int_X b_Q(y) h^\ell_{j,k}(y)\, d\mu(y)$ vanishes. This finishes the proof of (i).
(ii) The proof of (ii) follows from usual arguments of interpolation and duality since $T_{\tilde{\mathcal{A}}^F,\epsilon}$ is self adjoint.
(iii) Let Q be a fixed dyadic set in $\mathcal{D}$ and let us write $f = f_1 + f_2$ with $f_1 = f\chi_Q$. Let $x \in Q$, then

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |T_{\tilde{\mathcal{A}}^F,\epsilon}f(y) - T_{\tilde{\mathcal{A}}^F,\epsilon}f_2(x)|\, d\mu(y) &\leq \frac{1}{\mu(Q)} \int_Q |T_{\tilde{\mathcal{A}}^F,\epsilon}f_1(y)|\, d\mu(y) \\ &+ \frac{1}{\mu(Q)} \int_Q |T_{\tilde{\mathcal{A}}^F,\epsilon}f_2(y) - T_{\tilde{\mathcal{A}}^F,\epsilon}f_2(x)|\, d\mu(y) \\ &= I + II. \end{aligned}$$

Applying Hölder inequality and (ii) we get that

$$\begin{aligned} I &\leq \left(\frac{1}{\mu(Q)} \int_Q |T_{\tilde{\mathcal{A}}^F,\epsilon}f_1(y)|^s\, d\mu(y) \right)^{1/s} \\ &\leq \frac{C_s}{\mu(Q)^{1/s}} \left(\int_Q |f(y)|^s\, d\mu(y) \right)^{1/s} \\ &\leq C_s [M^{dy}(|f|^s)(x)]^{1/s}. \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} II \leq & \frac{1}{\mu(Q)} \sum_{(j,k)\in\tilde{\mathcal{A}}^F} \sum_{\ell\in L'(j,k)} \left(\int_X |f_2(z) h^\ell_{j,k}(z)|\, d\mu(z) \right) \\ & \times \left(\int_Q |h^\ell_{j,k}(x) - h^\ell_{j,k}(y)|\, d\mu(y) \right). \end{aligned} \tag{7.3}$$

Notice that the general term of the sum of the right hand side above vanishes. In fact, if $Q^j_k \subseteq Q$, since $f_2 = f\chi_{Q^c}$ we have that the first integral in the product is zero. If $Q \cap Q^j_k = \emptyset$, then, since $x \in Q$, hence $h^\ell_{j,k}(x) = 0$, so that by (h1) the second integral is zero. Finally, if $Q \subsetneq Q^j_k$, hence by construction of the Haar functions $h^\ell_{j,k}(x) = h^\ell_{j,k}(y)$ and the second integral is again equals to zero.
So that, (iii) follows by taking $a = T_{\tilde{\mathcal{A}}^F,\epsilon}f_2(x)$ in the definition of the sharp maximal function $M^{\#,dy}$. $\square$

The next lemma is nothing but an elementary extension of the quantitative information contained in the unconditionality of a basis for a Banach subspace of any L^p, essentially provided by Khintchine inequality (see for example Corollary 7.11 in [19]).

Lemma 7.3. *Let Y be a Banach subspace of some $L^p(d\nu)$ $(1 < p < \infty)$ with ν a σ–finite positive Borel measure on X. Let $\{\phi_n : n \in A\}$ be an unconditional basis for Y. Then, there exist constants $0 < c \leq C < \infty$ such that the inequalities*

$$c\Big\|\sum_{n\in A} a_n\phi_n\Big\|_{p,d\nu} \leq \Big\|\Big(\sum_{n\in A}|a_n|^2|\phi_n|^2\Big)^{1/2}\Big\|_{p,d\nu} \leq C\Big\|\sum_{n\in A} a_n\phi_n\Big\|_{p,d\nu},$$

hold for every sequence $\{a_n : n \in A\}$ of real numbers.

Proof of Theorem 7.1. Let us first show that $\mathcal{H}$ is an unconditional basis for $\mathcal{L}^p$ $(1 < p < \infty)$. This entangles to prove two basic facts about the system $\mathcal{H}$. First that the linear span of $\mathcal{H}$ is dense in each $\mathcal{L}^p$ $(1 < p < \infty)$. This fact follows from Lemma 5.2 and Theorem 6.2 in the following way. Given $f \in \mathcal{L}^p$ one can use Lemma 5.2 to approximate f in the L^p-norm by a simple function of dyadic sets g, with mean value zero if $\mu(X) < \infty$. Since g is simple on $\mathcal{D}$, we have that for some $j_o \in \mathbb{Z}$, $g \in V_{j_o}$ thus $g = P_{j_o}g$. Hence, from Theorem 6.2, g itself can be approximated in L^p by a function of the form $(P_{j_o} - P_j)g = \sum_{i=j}^{j_o-1}(P_{i+1} - P_i)g$ with $j < j_o$, which is a linear combination of elements of $\mathcal{H}$, since for each $i = j, \ldots, j_o - 1$ the set $\{k : Q^i_k \cap \{g \neq 0\} \neq \emptyset\}$ is finite, so that each $(P_{i+1} - P_i)g$ is also a finite linear combination of the set $\{h^\ell_{j,k} : k \in \mathcal{K}(i),\ \ell \in L'(i,k)\}$. Notice that, since for each $h \in \mathcal{H}$ the linear functional $h^*(f) = \langle f, h\rangle$ is continuous on $\mathcal{L}^p$, we have that $f = \sum_{h\in\mathcal{H}}\langle f, h\rangle h$ in the sense of L^p for every $f \in \mathcal{L}^p$. Second that there exists a constant C such that for every $f \in \mathcal{L}^p$, every sequence $\{\varepsilon(h) : h \in \mathcal{H}\}$ such that $\varepsilon(h) = \pm 1$, and every finite subset $\mathcal{H}'$ of $\mathcal{H}$ we have the inequality $\left\|\sum_{h\in\mathcal{H}'}\varepsilon(h)\langle f, h\rangle\, h\right\|_p \leq C\|f\|_p$. This fact follows from (ii) of Theorem 7.2. Finally the inequalities (7.1) follow from Lemma 7.3. □

8. Weighted inequalities for the projection operators

The purpose of this section is to obtain weighted results in the context of spaces of homogeneous type for the projection operators induced by a dyadic family $\mathcal{D}$ like the one defined in Section 2.

We shall say that a Borel measure ν is non-trivial if it is not identically zero neither identically ∞.

The results obtained in this section are contained in the next two theorems.

Theorem 8.1. *(The case $p > 1$) Let (X, d, μ) be a space of homogeneous type and $1 < p < \infty$. Let us consider $\{P_j\}_{j\in\mathbb{Z}}$, the sequence of projection operators defined in* (6.1) *and let ν be a non trivial positive regular Borel measure on X. Then, the following statements are equivalent:*

(A1) (i) *the operators P_j are continuous on $L^p(d\nu)$,*
(ii) *for every $f \in L^p(d\nu)$, $\lim_{j\to+\infty}\|f - P_jf\|_{p,d\nu} = 0$ and*
(iii) *if $\mu(X) = \infty$, $\lim_{j\to-\infty}\|Pjf\|_{p,d\nu} = 0$, for all $f \in L^p(d\nu)$;*

(A2) *the operators P_j are uniformly bounded on $L^p(d\nu)$;*

(A3) *the operators* P_j *are uniformly of weak type* (p,p) *with respect to* ν*;*
(A4) *the operator* P^* *defined in* (6.2) *is of weak type* (p,p) *with respect to* ν*;*
(A5) *the operator* P^*f *is of strong type* (p,p) *with respect to* ν*;*
(A6) ν *is absolutely continuous with respect to* μ *and* $d\nu = wd\mu$ *with* $w \in A_p^{dy}$.

Further, each one of the above statements implies

(A) *for* $f \in L^p(d\nu)$, $P_jf \to f$ *a.e. when* $j \to +\infty$ *and if* $\mu(X) = \infty$, $P_jf \to 0$ *a.e. when* $j \to -\infty$.

Theorem 8.2. *(The case* $p = 1$*) Let* (X, d, μ) *be a space of homogeneous type. Let us consider* $\{P_j\}_{j\in\mathbb{Z}}$*, the sequence of projection operators defined in* (6.1) *and let* ν *be a non trivial positive regular Borel measure on* X*. Then, the following statements are equivalent:*

(B1) *the operators* P_j *are uniformly bounded on* $L^1(d\nu)$*;*
(B2) *the operators* P_j *are uniformly of weak type* $(1,1)$ *with respect to* ν*;*
(B3) *the operator* P^*f *is of weak type* $(1,1)$ *with respect to* ν*;*
(B4) ν *is absolutely continuous with respect to* μ *and* $d\nu = wd\mu$ *with* $w \in A_1^{dy}$.

Further, each one of the above statements implies

(B) *for* $f \in L^1(d\nu)$, $P_jf \to f$ *a.e. and in* L^1 *norm, when* $j \to +\infty$ *and if* $\mu(X) = \infty$, $P_jf \to 0$ *a.e. when* $j \to -\infty$.

Proof of Theorems 8.1 and 8.2. Observe that the implications (A5)⇒ (A4)⇒ (A3), (A5)⇒ (A2)⇒ (A3), (B1)⇒ (B2) and (B3)⇒ (B2) are obvious. On the other hand, (A1)⇒ (A2) and (B4)⇒ (B1) follow as in the proof of Theorem 5 in [4]. We shall write with some detail the remaining implications since they are more related with the geometric nature of the current setting.
(A6)⇒ (A5) and (B4)⇒ (B3) follow from Theorem 4.1 (i) and (ii), since as we have already observed P^* is bounded by the dyadic Hardy-Littlewood maximal function. In order to show that (A3)⇒ (A6) and (B2)⇒ (B4) with the argument used in the proof of Theorem 5 in [4], we only have to observe that the exact covering obtained by our modification of Christ's construction given in Theorem 2.1 allows us to prove the absolute continuity of ν with respect to μ.
To prove that (A6)⇒ (A) observe that $L^p \cap L^p(w)$ is a dense subset of $L^p(w)$ since, following the argument in Lemma 5.2, it is easy to see that the linear span of the set $\{\chi_Q : Q \in \mathcal{D}\}$ is dense in each $L^p(w)$ with $p < \infty$. Then from Theorem 6.1, since (A6)⇒ (A5), we get that $P_jf \to f$ a.e. for $j \to +\infty$ and every $f \in L^p(w)$. If $\mu(X) = \infty$, using Theorem 6.2 (a), we get in a similar way that $P_jf \to 0$ a.e. for $j \to -\infty$ and every $f \in L^p(w)$.
Notice that (A5)⇒ (A1) follows from (A5)⇒ (A6)⇒ (A) and Lebesgue dominated convergence Theorem.
Finally, we can see that (B1) and (B4) imply (B) with the arguments used when $p > 1$. □

9. The Haar system as an unconditional basis for weighted L^p spaces

Let $\mathcal{D}$ be a family of dyadic sets as defined in Section 2 and let $\mathcal{H}$ be the associated Haar system introduced in Section 5. By $\tilde{\mathcal{H}}$ we shall denote the Haar basis $\mathcal{H}$ when $\mu(X) = \infty$ and $\mathcal{H} \cup \{(\mu(X))^{-1/2}\}$ when $\mu(X) < \infty$. The main result in this section is contained in the next statement.

Theorem 9.1. *Let (X, d, μ) be a space of homogeneous type and $1 < p < \infty$. Let ν be a non trivial positive regular Borel measure on X and finite on bounded Borel subsets of X. Then, the following statements are equivalent:*

(H1) *$\tilde{\mathcal{H}}$ is an unconditional basis for $L^p(d\nu)$ and the functionals $h^*(f) = \langle f, h\rangle = \int_X hf\, d\mu$ belong to the dual space of $L^p(d\nu)$ for each $h \in \tilde{\mathcal{H}}$;*

(H2) *ν is absolutely continuous with respect to μ and $d\nu = w d\mu$ with $w \in A_p^{dy}$;*

(H3) *$\nu(Q) > 0$ for every $Q \in \mathcal{D}$ and there exist two constants C_1 and C_2 such that*

$$C_1 \, \|f\|_{p,d\nu} \leq \left\| \left(\sum_{h\in\tilde{\mathcal{H}}} |\langle f, h\rangle|^2 |h|^2 \right)^{1/2} \right\|_{p,d\nu} \leq C_2 \|f\|_{p,d\nu}$$

hold for every $f \in L^p(d\nu)$.

In order to prove the above theorem we shall make use of the following result that proves the weighted uniform boundedness of the operators $T_{\tilde{\mathcal{A}}^F,\epsilon}$ defined in (7.2).

Theorem 9.2. *Let $1 < p < \infty$, $T_{\tilde{\mathcal{A}}^F,\epsilon}$ the operators defined in* (7.2) *and $w \in A_p^{dy}$. Then, there exists a constant $C > 0$ independent of $\tilde{\mathcal{A}}^F$ and ϵ such that*

$$\|T_{\tilde{\mathcal{A}}^F,\epsilon} f\|_{p,w} \leq C \|f\|_{p,w},$$

holds for every locally integrable function f.

Proof. First, observe that, since $\mathcal{H} \subset L^\infty$ we get that $T_{\tilde{\mathcal{A}}^F,\epsilon} f(x) \in L^p(w)$ for every $f \in L^p(w)$. This fact follows from the inequality

$$|T_{\tilde{\mathcal{A}}^F,\epsilon} f(x)| \leq C(\tilde{\mathcal{A}}^F) M^{dy} f(x),$$

where $C(\tilde{\mathcal{A}}^F)$ is a finite constant that depends on $\tilde{\mathcal{A}}^F$. Now, since $w \in A_p^{dy} \subset A_\infty^{dy}$, applying Theorem 4.1 (iii) to the functions $T_{\tilde{\mathcal{A}}^F,\epsilon} f$ we get that

$$\begin{aligned} \int_X |T_{\tilde{\mathcal{A}}^F,\epsilon} f|^p w \, d\mu &\leq \int_X [M^{dy}(T_{\tilde{\mathcal{A}}^F,\epsilon} f)]^p w \, d\mu \\ &\leq C w(X) [m_X(|T_{\tilde{\mathcal{A}}^F,\epsilon} f|)]^p + C \int_X [M^{\#,dy}(T_{\tilde{\mathcal{A}}^F,\epsilon} f)]^p w \, d\mu. \end{aligned} \tag{9.1}$$

Notice that, by Theorem 7.2 (ii), for every $s > 1$ we have

$$\begin{aligned} [m_X(|T_{\tilde{\mathcal{A}}^F,\epsilon} f|)]^s &\leq \frac{1}{\mu(X)} \int_X |T_{\tilde{\mathcal{A}}^F,\epsilon} f|^s \, d\mu \\ &\leq \frac{C_s^s}{\mu(X)} \int_X |f|^s \, d\mu = C_s^s m_X(|f|^s). \end{aligned}$$

Then, since the term $Cw(X)[m_X(|T_{\tilde{\mathcal{A}}^F,\epsilon} f|)]^p$ only appears when $\mu(X) < \infty$ and in this case there exists $Q \in \mathcal{D}$ such that $X = Q$, from the above inequalities we have

$$\begin{aligned} w(X)[m_X(|T_{\tilde{\mathcal{A}}^F,\epsilon} f|)]^p &\leq C_s^p \int_X [m_X(|f|^s)]^{p/s} w \, d\mu \\ &\leq C_s^p \int_X [M^{dy}(|f|^s)]^{p/s} w \, d\mu. \end{aligned}$$

Now, by using the above inequalities and Theorem 7.2 (iii) in (9.1) we get that

$$\int_X |T_{\tilde{\mathcal{A}}^F,\epsilon} f|^p w\, d\mu \le C \int_X [M^{dy}(|f|^s)]^{p/s} w\, d\mu,$$

where C depend on s and p. Finally, notice that for each $w \in A_p^{dy}$ there is an s bigger than one such that $p/s > 1$ and $w \in A_{p/s}^{dy}$ (see Theorem 5.1 in [3]). Then, the theorem follows from this fact applying Theorem 4.1 (ii). □

Proof of Theorem 9.1. Let us first prove that (H1) and (H2) are equivalent. In order to show that (H2) ⇒ (H1) we begin checking the $L^p(w)$–continuity of the linear functional $h^*(f) = \langle f, h\rangle$ for every $h \in \tilde{\mathcal{H}}$. This fact follows easily from Hölder's inequality,

$$\begin{aligned} |h^*(f)| &\le \int |f|\ |h| w^{1/p} w^{-1/p} d\mu \\ &\le \left(\int |f|^p\ w\ d\mu\right)^{1/p} \|h\|_\infty\ \sigma(Q(h))^{\frac{p-1}{p}}, \end{aligned}$$

since $\sigma(Q) = \int_Q w^{-\frac{1}{p-1}}$ is finite for every $Q \in \mathcal{D}$ and, of course, so is $\|h\|_\infty$. Here $Q(h)$ is the supporting dyadic set for h. Now, we shall prove the density of the linear span of $\tilde{\mathcal{H}}$ in $L^p(w)$. Actually this density property is a consequence of a weighted version of Lemma 5.2 and Theorem 8.1. In fact, given $f \in L^p(w)$ and $\epsilon > 0$, there exists g a simple function of dyadic sets such that $\|f - g\|_{p,w} < \epsilon$ and such a g must belong to some V_{j_o}. If $\mu(X) < \infty$, since the constant functions belong to the linear span of $\tilde{\mathcal{H}}$, g itself is in the linear span of $\tilde{\mathcal{H}}$. On the other hand, if $\mu(X) = \infty$, since for $j < j_o$, $g - P_j g = (P_{j_o} - P_j)g$ is in the linear span of $\tilde{\mathcal{H}}$ and we have that $\|f - (g - P_j g)\|_{p,w} \le \|f - g\|_{p,w} + \|P_j g\|_{p,w} < \epsilon + \|P_j g\|_{p,w}$, the density property follows from (A6)⇒ (A1) in Theorem 8.1, because we can make $\|P_j g\|_{p,w}$ as small as desired by taking j small enough. Notice that from the facts proved above we have that for every $f \in L^p(w)$ we can write $f = \sum_{h\in\tilde{\mathcal{H}}} \langle f, h\rangle h$ in the sense of $L^p(w)$. Finally, the unconditionality of $\tilde{\mathcal{H}}$ follows from Theorem 9.2.
For the proof of (H1)⇒ (H2) we shall closely follow the lines of the proofs in [4], nevertheless, attending to the geometric diversity of our current setting, we shall make it self–contained. Let us prove (H1)⇒ (H2) in five steps, the first two are devoted to show that ν is absolutely continuous with respect to μ.
Step a: Since in (H1) we are assuming that $\tilde{\mathcal{H}}$ is an unconditional basis for $L^p(d\nu)$, for every $f \in L^p(d\nu)$ there must exist a scalar sequence $\{\alpha_h : h \in \tilde{\mathcal{H}}\}$ such that $f = \sum_{h\in\tilde{\mathcal{H}}} \alpha_h h$ in the sense of $L^p(d\nu)$. Since, on the other hand, h^* is linear and continuous on $L^p(d\nu)$ we have that $\alpha_h = \langle f, h\rangle$.
Step b: (H1) implies that for a given Borel set A in X, $\mu(A) = 0$ if and only if $\nu(A) = 0$. In other words μ and ν are equivalent measures. In fact, assume first that A is a bounded Borel set with $\mu(A) = 0$. Then $\chi_A \in L^p(d\nu)$ and from **Step a** we have that $\chi_A = \sum_{h\in\tilde{\mathcal{H}}} \langle \chi_A, h\rangle\, h$ in the $L^p(d\nu)$ sense. On the other hand $\langle \chi_A, h\rangle = \int_A h\, d\mu = 0$ for every $h \in \tilde{\mathcal{H}}$. Hence $\chi_A = 0$ ν-almost everywhere and $\nu(A) = 0$. Let us take now A a bounded Borel subset of X with $\nu(A) = 0$. Hence $\chi_A \in L^p(d\nu)$ and $\|\chi_A\|_{p,d\nu} = 0$. Since each h^* is continuous on $L^p(d\nu)$ we have, for some constant C_h, that $|\langle \chi_A, h\rangle| = |h^*(\chi_A)| \le C_h \|\chi_A\|_{p,d\nu} = 0$, for every $h \in \tilde{\mathcal{H}}$. Since χ_A belongs also to L^2, because the finiteness of the μ-measure of the balls, from (5.1) in the case of infinite measure and (5.3) in the case of finite measure, we

have that $\chi_A = \sum_{h\in\tilde{\mathcal{H}}}\langle \chi_A, h\rangle\, h$ in the L^2 sense. But, as we noticed, $\langle \chi_A, h\rangle = 0$ for every $h \in \tilde{\mathcal{H}}$, hence $\chi_A = 0$ in L^2, or $\mu(A) = 0$. For general A, non-necessarily bounded, the equivalence of μ and ν follows from the σ-finiteness of both measures. Set $w = \frac{d\nu}{d\mu}$.

Step c: Let $j_0 \in \{-\infty\} \cup \mathbb{Z}$ be the largest integer for which $X \in \mathcal{D}_{j_0}$. Of course, $j_0 = -\infty$ if and only if $\mu(X) = \infty$. For a given $h \in \tilde{\mathcal{H}}$ we shall write $Q(h)$ to denote the dyadic set in $\mathcal{D}$ supporting h and $j(h)$ will be an integer such that $Q(h) \in \mathcal{D}_{j(h)}$. For $m \in \mathbb{Z}$ and $m > j_0$, (H1) implies that the operators $S_m f = \sum_{j(h)\le m}\langle f, h\rangle\, h = \sum_{h\in\tilde{\mathcal{H}}} \beta_h\langle f, h\rangle\, h$, with $\beta_h = 1$ is $j(h) \le m$ and $\beta_h = 0$ if $j(h) > m$, are uniformly bounded on $L^p(d\nu)$. In other words $\sup_m \|S_m f\|_{p,d\nu} \le C\|f\|_{p,d\nu}$.

Step d: Notice that since μ and ν are equivalent, then the spaces of essentially bounded functions $L^\infty(d\nu)$ and $L^\infty(d\mu)$ coincide. Let us denote by L^∞_b the space of essentially bounded functions which vanishes outside a bounded set. Take now $f \in L^\infty_b$. Since $f \in L^p(d\nu)$, we have that $S_m f = \sum_{\{h\in\tilde{\mathcal{H}}: j(h)\le m\}}\langle f, h\rangle\, h$ in the $L^p(d\nu)$ sense. On the other hand, since $f \in L^2$ and $V_m = \oplus_{j\le m} W_j$ if $\mu(X) = \infty$ and $V_m = V_{j_0} \oplus (\oplus_{j_0\le j\le m} W_j)$ if $\mu(X) < \infty$, we have that $P_m f = \sum_{\{h\in\tilde{\mathcal{H}}: j(h)\le m\}}\langle f, h\rangle\, h$ in the L^2 sense. Hence, except for a Borel set of the form $N = N_1 \cup N_2$ with $\nu(N_1) = 0 = \mu(N_2)$ we have that $S_m f(x) = P_m f(x)$. Then $\|P_m f\|_{p,d\nu} = \|S_m f\|_{p,d\nu} \le C\|f\|_{p,d\nu}$, for every $f \in L^\infty_b$.

Step e: We shall show that $w = \frac{d\nu}{d\mu}$ belongs to A^{dy}_p as in the proof of Theorem 5 in [4]. Let $Q \in \mathcal{D}_j$ fixed, $\varepsilon > 0$ and $\sigma_\varepsilon = (w + \varepsilon)^{-\frac{1}{p-1}}$. Observe that, for all $x \in Q$ we have that

$$P_j(\chi_Q \sigma_\varepsilon)(x) = \frac{1}{\mu(Q)}\int_Q \sigma_\varepsilon \, d\mu.$$

From Chebyshev's inequality and **Step d**, since $\chi_Q\sigma_\varepsilon \in L^\infty_b$, we have that

$$\begin{aligned} w(Q) &\le w\left(\left\{P_j(\chi_Q\sigma_\varepsilon)(x) > \frac{\sigma_\varepsilon(Q)}{2\mu(Q)}\right\}\right) \\ &\le \left(\frac{2\mu(Q)}{\sigma_\varepsilon(Q)}\right)^p \int_X |P_j(\chi_Q\sigma_\varepsilon)|^p w\, d\mu \\ &\le C\left(\frac{\mu(Q)}{\sigma_\varepsilon(Q)}\right)^p \int_Q \sigma_\varepsilon^p w\, d\mu, \end{aligned}$$

which gives A^{dy}_p for $\varepsilon \to 0$.

To prove that (H2), or (H1), implies (H3), notice first that $\nu(Q) > 0$ for every $Q \in \mathcal{D}$ since $d\nu = w d\mu$ with $w \in A^{dy}_p$. Finally, the inequalities in (H3) follow from (H1) by applying Lemma 7.3.

Let us finally sketch how (H3) implies (H1) following the lines of [11]. Notice first that each h in $\tilde{\mathcal{H}}$ belongs to every $L^p(d\nu)$, $1 \le p \le \infty$, since actually h is bounded and vanishes outside a bounded set. Let us now prove that, from the right hand side inequality in (H3) and from the fact that $\nu(Q) > 0$ for every $Q \in \mathcal{D}$, we get that h^* is bounded functional on $L^p(d\nu)$, for every $h \in \tilde{\mathcal{H}}$. In fact

$$\begin{aligned} |\langle f, h\rangle|\, \|h\|_{p,d\nu} &\le \left\| \Big(\sum_{h\in\tilde{\mathcal{H}}} |\langle f, h\rangle|^2\, |h|^2\Big)^{1/2} \right\|_{p,d\nu} \\ &\le C_2\|f\|_{p,d\nu}. \end{aligned}$$

It remains to prove that $\tilde{\mathcal{H}}$ is an unconditional basis for $L^p(d\nu)$. Given any finite subset $\mathcal{H}'$ of $\tilde{\mathcal{H}}$ let us consider the operator $T_{\mathcal{H}'}f = \sum_{h\in\mathcal{H}'}\langle f, h\rangle h$, which is defined for $f \in L^p(d\nu)$, since the sum is finite and we know that each h^* belongs to the dual of $L^p(d\nu)$. Moreover $T_{\mathcal{H}'}f \in L^p(d\nu)$ and belongs to the linear span of $\tilde{\mathcal{H}}$. Let us prove that $T_{\mathcal{H}'}f$ can be chosen as close as desired to f by taking $\mathcal{H}'$ large enough but finite. From the first inequality in (H3) applied to $f - T_{\mathcal{H}'}f$ with $f \in L^p(d\nu)$ we have that

$$\begin{aligned} C_1\big\|f - T_{\mathcal{H}'}f\big\|_{p,d\nu} &\le \Big\|\Big(\sum_{h\in\tilde{\mathcal{H}}} |\langle f - T_{\mathcal{H}'}f, h\rangle|^2\,|h|^2\Big)^{1/2}\Big\|_{p,d\nu} \\ &\le \Big\|\Big(\sum_{h\notin\mathcal{H}'} |\langle f, h\rangle|^2\,|h|^2\Big)^{1/2}\Big\|_{p,d\nu}. \end{aligned} \tag{9.2}$$

Now, since $\big\|\big(\sum_{h\in\tilde{\mathcal{H}}}|\langle f, h\rangle|^2\,|h|^2\big)^{1/2}\big\|_{p,d\nu} \le C_2\|f\|_{p,d\nu} < \infty$, the right hand side in (9.2) tends to zero as $\mathcal{H}'$ tends to cover $\tilde{\mathcal{H}}$. Hence the $L^p(d\nu)$–closure of the linear span of $\tilde{\mathcal{H}}$ coincide with $L^p(d\nu)$. On the other hand, if $\{\varepsilon(h) : h \in \mathcal{H}\}$ is any sequence such that $\varepsilon(h) \pm 1$, for any finite subset $\mathcal{H}'$ of $\tilde{\mathcal{H}}$ we have that

$$\begin{aligned} \Big\|\sum_{h\in\mathcal{H}'}\varepsilon(h)\langle f, h\rangle\, h\Big\|_{p,d\nu} &\le \frac{1}{C_1}\Big\|\Big(\sum_{h\in\mathcal{H}'}|\langle f, h\rangle|^2\,|h|^2\Big)^{1/2}\Big\|_{p,d\nu} \\ &\le \frac{1}{C_1}\Big\|\Big(\sum_{h\in\tilde{\mathcal{H}}}|\langle f, h\rangle|^2\,|h|^2\Big)^{1/2}\Big\|_{p,d\nu} \\ &\le \frac{C_2}{C_1}\|f\|_{p,d\nu}, \end{aligned}$$

which finishes the proof of the theorem. □

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