COMPOSITION OF FRACTIONAL ORLICZ MAXIMAL OPERATORS AND A_1–WEIGHTS ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. For a Young function Θ and $0 \leq \alpha < 1$, let $M_{\alpha,\Theta}$ be the fractional Orlicz maximal operator defined in the context of the spaces of homogeneous type (X, d, μ) by $M_{\alpha,\Theta}f(x) = \sup_{x\in B} \mu(B)^{\alpha}||f||_{\Theta,B}$, where $||f||_{\Theta,B}$ is the mean Luxemburg norm of f on a ball B. When $\alpha = 0$ we simply denote it by M_Θ. In this paper we prove that if Φ and Ψ are two Young functions, there exists a third Young function Θ such that the composition $M_{\alpha,\Psi} \circ M_\Phi$ is pointwise equivalent to $M_{\alpha,\Theta}$. As a consequence we prove that for some Young functions Θ, if $M_{\alpha,\Theta}f < \infty$ a.e. and $\delta \in (0, 1)$ then $(M_{\alpha,\Theta}f)^{\delta}$ is an A_1-weight.

1. Introduction

Let us consider a space of homogeneous type (X, d, μ), that is, X is a set endowed with a quasi-distance d such that the balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are open sets, and with a positive measure μ satisfying a doubling condition (we refer Section 2 for a more complete definition). Given a locally integrable function f on X, let $M_\alpha f$, $0 \leq \alpha < 1$, be the fractional maximal operator defined by

$$M_\alpha f(x) = \sup_{x\in B} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls B containing x. If $\alpha = 0$ we get the Hardy-Littlewood maximal operator; in this case, we drop the subscript α.

It is known that the following result holds for $\mathcal{M} = M_\alpha$:

$$\text{If } \mathcal{M}f < \infty \text{ a.e. and if } \delta \in (0, 1), \text{ then } (\mathcal{M}f)^{\delta} \in A_1, \tag{1.1}$$

where A_1 is the Muckenhoupt class of nonnegative locally integrable functions w such that

$$\mathrm{A}_1 : \qquad \frac{1}{\mu(B)} \int_B w\, d\mu \leq Cw(x), \quad \text{a.e. } x \in B, \tag{1.2}$$

for all balls B. The proof of this result follows by standard arguments (see [11] for the euclidean case and [5] for the case $\alpha = 0$), that is, if $\tilde{B}$ is a suitable dilation of B, writing $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$, it is enough to prove that (1.2) holds with w replaced by each $(M_\alpha f_i)^{\delta}$, $i = 1, 2$. To establish (1.2) for $(M_\alpha f_1)^{\delta}$ the weak $(1, \frac{1}{1-\alpha})$

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type inequality of M_α is applied and, for the other case, the fact that for any two points x, y belonging B we have

$$M_\alpha(f_2)(y) \leq C\ M_\alpha(f_2)(x). \tag{1.3}$$

For $0 \leq \alpha < 1$, a generalization of the operator M_α is the fractional Orlicz maximal operator associated to a Young function Φ defined, for each function f on X, by

$$M_{\alpha,\Phi}f(x) = \sup_{x\in B} \mu(B)^\alpha \|f\|_{\Phi,B},$$

where the supremum is taken over all balls B containing x and

$$||f||_{\Phi,B} = \inf\left\{\lambda > 0 : \frac{1}{\mu(B)}\int_B \Phi\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) \leq 1\right\} \tag{1.4}$$

is the Φ-mean Luxemburg norm of a function f on a ball B. When $\alpha = 0$ we also drop the subscript α. When $\Phi(t) = t$, $M_{\alpha,\Phi}$ is the fractional maximal operator M_α.

In the last years, weighted inequalities with non-a-priori assumption on the weights have been proved for linear operators like singular integrals, fractional integrals and their commutators by using duality arguments (for the euclidean setting and $\alpha = 0$ see for example [12] and [13], for spaces of homogeneous type see [15] for the case $\alpha = 0$ and [3] for $0 \leq \alpha < 1$). One of the main tools in the proofs of these inequalities is to establish (1.1) for $\mathcal{M} = M_{\alpha,\Phi}$ with suitable Young functions Φ. If $\alpha = 0$ then (1.1) can be proved for $\mathcal{M} = M_\Phi$ and any Young function by using the same arguments described above. However, it is not clear how to prove (1.1) for $\mathcal{M} = M_{\alpha,\Phi}$ and $\alpha \neq 0$ by applying the standard arguments, although it is possible to obtain a result like (1.3) for $M_{\alpha,\Phi}$ (see Lemma 4.2 in [3]) and there is an end-point estimate for this operator for some Young functions (see [7] and [8]). We point out that in [3] the authors proved that (1.1) holds for $\mathcal{M} = M_{\alpha,\Phi_k}$ in the special case $\Phi_k(t) = t[\log(e+t)]^k$, $k \in \mathbb{N}$. The proof of this result is based on the fact that M_{α,Φ_k} is equivalent to the composition $M_\alpha(M^k)$ where M^k is the Hardy-Littlewood maximal operator iterated k times.

One of the purposes of this paper is to prove that (1.1) holds for the maximal operators $M_{\alpha,\Phi}$ for more general Young functions Φ. This result will be a consequence of the following type of result: given two Young functions Ψ and Φ, we shall define a third Young function Θ, such that the composition $M_{\alpha,\Psi} \circ M_\Phi$ is equivalent to the operator $M_{\alpha,\Theta}$. The proof of this last result will be the main purpose of this article.

Before stating the theorem we shall observe some properties of the maximal functions $M_{\alpha,\Phi}$. Let Φ_1 and Φ_2 be Young functions. We say that Φ_2 dominates Φ_1 at ∞, and denote it by $\Phi_1 \prec_\infty \Phi_2$, if there exist $a, b, t_0 > 0$ such that

$$\Phi_1(t) \leq b\Phi_2(at) \qquad \text{for all } t \geq t_0.$$

If $\Phi_1 \prec_\infty \Phi_2$ then there exists a constant C, depending on Φ_1 and Φ_2, such that $||f||_{\Phi_1,B} \leq C||f||_{\Phi_2,B}$ for all balls B and functions f. Since the constant C is independent of B, we get that $M_{\alpha,\Phi_1}f(x) \leq CM_{\alpha,\Phi_2}f(x)$. We say that Φ_1 is equivalent to Φ_2 at ∞, and denote it by $\Phi_1 \approx_\infty \Phi_2$, if $\Phi_1 \prec_\infty \Phi_2$ and $\Phi_2 \prec_\infty \Phi_1$. Therefore, if $\Phi_1 \approx_\infty \Phi_2$ then $M_{\alpha,\Phi_1} \approx M_{\alpha,\Phi_2}$, that is, there exist two positive constants C_1 and C_2

such that $C_1 M_{\alpha,\Phi_1} f(x) \leq M_{\alpha,\Phi_2} f(x) \leq C_2 M_{\alpha,\Phi_1} f(x)$.

Let Φ be a Young function and define

$$\Phi_0(t) = \begin{cases} 0 & \text{if } \ 0 \leq t < 1 \\ \Phi(t) - \Phi(1) & \text{if } \ t \geq 1. \end{cases} \tag{1.5}$$

Since Φ_0 is a Young function and $\Phi_0 \approx_\infty \Phi$, it is clear that $M_{\alpha,\Phi_0} \approx M_{\alpha,\Phi}$.

Now we are ready to state our main result.

Theorem 1.1. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$. Let Ψ and Φ be two Young functions and $0 \leq \alpha < 1$. We define the function*

$$\Theta(t) = \int_0^t \Psi_0'(u)\Phi(t/u)\,du, \tag{1.6}$$

where Ψ_0 is defined as in (1.5) and Ψ_0' is the derivative of Ψ_0. Then, Θ is a Young function and for all Young functions $\bar{\Theta} \approx_\infty \Theta$ we get that

$$M_{\alpha,\bar{\Theta}} \approx M_{\alpha,\Psi}(M_\Phi). \tag{1.7}$$

When $X = \mathbb{R}^n$, d is the euclidean distance, μ is the Lebesgue measure and $\alpha = 0$ the equivalence (1.7) was proved in [4]. As far as we know, the result for the case $\alpha \neq 0$ is new even in the euclidean case.

Now, we shall show some examples. We introduce the following notation: if $\Phi(t) = t^r$ or $\Phi(t) = t^r(1 + \log^+ t)^\beta$, the fractional Orlicz maximal operators $M_{\alpha,\Phi}$ are respectively written as M_{α,L^r} and $M_{\alpha,L^r\,(\log L)^\beta}$; if $\alpha = 0$ we simply write M_{L^r} and $M_{L^r\,(\log L)^\beta}$.

Example 1: Let $p \geq 1$. Applying Theorem 1.1 with $\Psi(t) = t^p$ and $\Phi(t) = t^p(1 + \log^+ t)^\beta$, $\beta \geq 0$, we get

$$M_{\alpha,L^p}(M_{L^p(\log L)^\beta}) \approx M_{\alpha,L^p(\log L)^{\beta+1}}.$$

Notice that if $p = 1$ we get

$$M_\alpha(M_{L(\log L)^\beta}) \approx M_{\alpha,L(\log L)^{\beta+1}}. \tag{1.8}$$

In particular, when $p = 1$ and $\beta = 0$ we get that $M_\alpha(M) \approx M_{\alpha,L(\log L)}$. By induction, using (1.8) and the induction hypothesis with $\alpha = 0$, we easily obtain the known result

$$M_\alpha(M^k) \approx M_{\alpha,L(\log L)^k}, \quad k \in \mathbb{N},$$

where M^k is the iteration of the Hardy-Littlewood maximal operator k times (see Lemma 4.1 in [3]).

Example 2: If $\Psi(t) = t^p$ and $\Phi(t) = t^q$, $p, q \geq 1$ and $p \neq q$, then

$$M_{\alpha,L^p}(M_{L^q}) \approx M_{\alpha,L^r},$$

where $r = \max\{p, q\}$. In particular, for $p > 1$, $M_\alpha(M_{L^p}) \approx M_{\alpha,L^p}$.

Example 3: If $\Psi(t) = t$ and $\Phi(t) = t^p(1 + \log^+ t)^k$, with $k \in \mathbb{N}$ and $p > 1$, applying Theorem 1.1 we obtain

$$M_\alpha(M_{L^p(\log L)^k}) \approx M_{\alpha, L^p(\log L)^k}.$$

Example 4: From Example 3 and Example 1 (with $\alpha = 0$) we get

$$M_\alpha((M_{L^p})^{k+1}) \approx M_{\alpha, L^p(\log L)^k}, \quad k \in \mathbb{N} \text{ and } p > 1.$$

Notice that if we take $\Psi(t) = t$ in (1.6) we get that for $t > 1$

$$\Theta(t) = \int_1^t \Phi(t/u)\, du = t \int_1^t \Phi(u) u^{-2}\, du,$$

or equivalently $\Phi(t) = t\Theta'(t) - \Theta(t)$ for all $t > 1$. Then, it is easy to prove the following corollary of Theorem 1.1.

Corollary 1.2. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ and let $0 \le \alpha < 1$. Let Θ be a Young function which is not equivalent at ∞ to $\eta(t) = t$ and such that there exists a Young function Φ with $\Phi(t) = t\Theta'(t) - \Theta(t)$, for $t > 1$. Then, if $M_{\alpha,\Theta} f < \infty$ a.e. and $\delta \in (0, 1)$, we get that $(M_{\alpha,\Theta} f)^\delta \in A_1$.*

In fact, by Theorem 1.1 we get that $(M_{\alpha,\Theta} f)^\delta \approx [(M_\alpha(M_\Phi f)]^\delta$ and Corollary follows by (1.1) with $\mathcal{M} = M_\alpha$.

Remark 1.3. *Observe that if $\Theta \approx_\infty \eta$ with $\eta(t) = t$, then $M_{\alpha,\Theta} \approx M_\alpha$ and the corollary follows by standard arguments.*

Remark 1.4. *From the above examples we have that (1.1) holds for $\mathcal{M} = M_{\alpha,\Theta}$, where $\Theta(t) = t^p(1 + \log^+ t)^\beta$ for any $p \ge 1$ and $\beta \ge 1$.*

The article is organized in the following way: in Section 2 we give some preliminaries results and we prove a reverse inequality of the weak type inequality for the operator M_Φ, while Section 3 is devoted to prove Theorem 1.1.

2. Preliminaries and previous results

Given a set X, a function $d : X \times X \to \mathbb{R}_0^+$ is called a quasi-distance on X if the following conditions are satisfied:
(i) for every x and y in X, $d(x, y) \ge 0$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) for every x and y in X, $d(x, y) = d(y, x)$,
(iii) there exists a constant $K \ge 1$ such that

$$d(x, y) \le K(d(x, z) + d(z, y)) \tag{2.1}$$

for every x, y and z in X. We shall say that two quasi-distances d and d' on X are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 d'(x, y) \le d(x, y) \le c_2 d'(x, y)$ for all $x, y \in X$. In particular equivalent quasi-distances induce the same topology on X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x,r)=\{y : d(x,y)<r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0<\mu(B(x,2r))\leq A\mu(B(x,r))<\infty \tag{2.2}$$

holds for all $x\in X$ and $r>0$.

A structure (X,d,μ), with d and μ as above, is called a *space of homogeneous type*. The constants K and A in (2.1) and (2.2) will be called the constants of the space.

The balls in a general space of homogeneous type are not necessarily open. Macías and Segovia in [9] proved that there exists a continuous quasi-distance d' equivalent to d, for which every ball is open. In this article we always assume that the quasi-distance d is continuous and the balls are open sets. For a given quasi-distance d, sometimes we write $B_d(x,R)$ to describe the ball centred at x with radious R associated to d.

We shall say that a function $\Phi:[0,\infty)\to[0,\infty)$ is a Young function if there is a nontrivial, non-negative and increasing function ϕ such that $\Phi(t)=\int_0^t\phi(u)\,du$. Then, Φ is continuous, convex, increasing and satisfies $\Phi(0)=0$ and $\Phi(t)\to\infty$ as $t\to\infty$. For a Young function Φ, the maximal operator M_Φ satisfies the following weak type inequality

$$\mu(\{x\in X : M_\Phi f(x)>\lambda\})\leq C\int_X\Phi\left(\frac{|f(x)|}{\lambda}\right)d\mu(x). \tag{2.3}$$

The proof of the above inequality is similar to that of the $(1,1)$-weak type inequality for the Hardy Littlewood maximal operator (see [6]). By standard arguments, it follows from (2.3) that

$$\mu(\{x\in X : M_\Phi f(x)>\lambda\})\leq C\int_{\{x\in X:|f(x)|>\lambda/2\}}\Phi\left(\frac{2|f(x)|}{\lambda}\right)d\mu(x), \tag{2.4}$$

for some constant C, all $\lambda>0$ and all measurable function f.

In order to prove a suitable reverse inequality of (2.4) we shall need two results. The first one is a Calderón-Zygmung decomposition with Orlicz norms on a bounded space of homogeneous type. The proof follows the same steps as the one in [1] for the case $\Phi(t)=t$, so we omit it.

Lemma 2.1. *Let (X,d,μ) be a bounded space of homogeneous type, Φ a Young function and f a nonnegative function defined on X. Then, for every $\lambda>||f||_{\Phi,X}$, there exists a sequence of disjoint balls, $\{B_i\}=\{B(x_i,r_i)\}$ such that, if $\tilde{B}_i=B(x_i,Cr_i)$, where C is a constant depending only on the constant K in* (2.1)*, then*

(i) $||f||_{\Phi,\tilde{B}_i}\leq\lambda<||f||_{\Phi,B_i}$ *and*

(ii) $||f||_{\Phi,B}\leq\lambda$ *for every ball B centered at $x\in X\setminus\cup_i\tilde{B}_i$.*

Remark 2.2. *If (X,d,μ) is a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ we can apply the Lebesgue Differentiation Theorem in (ii) of Lemma 2.1 to obtain that $\Phi(f(x)/\lambda)\leq 1$ for almost every $x\in X\setminus\cup_i\tilde{B}_i$.*

The second result that we shall be dealing with is the following theorem due to Macías and Segovia ([10]).

Theorem 2.3. [10] *Let (X, d, μ) be a space of homogeneous type. There exists a quasi-distance δ on X which is equivalent to d such that, for some constant $C > 0$ depending only on the constants of the space, if $x \in X$, $0 < r \leq 6K^3R$ and $y \in B_\delta(x, R)$ then*

$$\mu(B_\delta(y, r) \cap B_\delta(x, R)) \geq C\mu(B_\delta(y, r)). \tag{2.5}$$

Moreover,

$$\delta(x, y) \leq d(x, y) \leq 3K^2\delta(x, y), \tag{2.6}$$

for every x and y in X.
The balls $B_\delta(x, R)$ endowed with the restrictions of the quasi-distance δ and the measure μ become bounded spaces of homogeneous type with constants K' and A', satisfying (2.1) and (2.2) respectively, independent of $R > 0$ and $x \in X$.

The above result provides us a quasi-distance δ equivalent to the quasi-distance d of the space with the property that the balls B_δ are spaces of homogeneous type. This property is not necesarily true for the balls B_d.

In the following lemma we state and prove a version of the reverse inequality for M_Φ.

Lemma 2.4. *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ and let δ be the quasi-distance defined in Theorem 2.3. Let $B_\delta = B_\delta(x, R)$ a fixed ball on X. Then, there exist positive constants C and D, depending only on the constants of the space, such that*

$$\int_{\left\{y\in B_\delta : \Phi\left(\frac{f(y)}{\lambda}\right)>1\right\}} \Phi\left(\frac{f(y)}{\lambda}\right) d\mu(y) \leq C\mu(\{y \in B_\delta : M_\Phi f(y) > D\lambda\}),$$

for any $\lambda > ||f||_{\Phi,B_\delta}$ and all non-negative functions f.

Proof. Given a non-negative function f on B_δ and $\lambda > ||f||_{\Phi,B_\delta}$, we apply Lemma 2.1 (and the corresponding Remark 2.2) to f at the level λ on the space of homogeneous type (B_δ, δ, μ). That is, there exists a sequence $\{x_i\} \subset B_\delta$ and disjoint δ-balls $S_i = B_\delta(x_i, r_i) \cap B_\delta$ in this space such that if $\widetilde{S}_i = B_\delta(x_i, Cr_i) \cap B_\delta$ with C depending only on K, then

(a) $||f||_{\Phi,\widetilde{S}_i} \leq \lambda < ||f||_{\Phi,S_i}$ and

(b) $\Phi\left(\frac{f(x)}{\lambda}\right) \leq 1$ for almost every $x \in B_\delta \setminus \cup_i \widetilde{S}_i$.

We start proving that there exists $D > 0$ such that for all i,

$$S_i \subset \{y \in B_\delta : M_\Phi f(y) > D\lambda\}. \tag{2.7}$$

Notice that, by (2.5) and (2.6) in Theorem 2.3 we get that $\mu(S_i) \geq C_1\mu(B_\delta(x_i, r_i))$, with $C_1 < 1$ and $B_\delta(x_i, r_i) \subset B_d(x_i, 3K^2r_i) \subset B_\delta(x_i, 3K^2r_i)$ respectively. Then $\mu(B_d(x_i, 3K^2r_i) \leq C_2\mu(B_\delta(x_i, r_i))$, with $C_2 > 1$. Now, since Φ is a convex function

$$\begin{aligned} \frac{1}{\mu(S_i)}\int_{S_i} \Phi\left(\frac{f}{\lambda}\right) d\mu &\leq \frac{1}{C_1\mu(B_\delta(x_i, r_i))}\int_{B_\delta(x_i,r_i)} \Phi\left(\frac{f}{\lambda}\right) d\mu \\ &\leq \frac{C_2}{C_1\mu(B_d(x_i, 3K^2r_i))}\int_{B_d(x_i,3K^2r_i)} \Phi\left(\frac{f}{\lambda}\right) d\mu \end{aligned}$$

$$\leq \frac{1}{\mu(B_d(x_i, 3K^2r_i))}\int_{B_d(x_i,3K^2r_i)} \Phi\left(\frac{C_2 f}{C_1\lambda}\right) d\mu.$$

Then, taking $D = \frac{C_1}{C_2}$ and using item (a) we get

$$\lambda < ||f||_{\Phi,S_i} \leq D^{-1}||f||_{\Phi,B_d(x_i,3K^2r_i)} \leq D^{-1}M_\Phi f(y),$$

for each $y \in S_i$, and we get (2.7). On the other hand, by item (b) we get

$$\mu\left(\left\{y \in B_\delta : \Phi\left(\frac{f(y)}{\lambda}\right) > 1\right\}\right) \leq \mu\left(\bigcup_i \widetilde{S}_i\right). \tag{2.8}$$

Finally, by (2.7), (2.8), (a) and (b) we get

$$\begin{aligned}
\mu(\{y \in B_\delta : M_\Phi f(y) > D\lambda\}) &\geq \sum_i \mu(S_i) \geq C\sum_i \mu(\widetilde{S}_i)\\
&\geq \sum_i \int_{\widetilde{S}_i} \Phi\left(\frac{f}{\lambda}\right) d\mu \geq \int_{\cup_i \widetilde{S}_i} \Phi\left(\frac{f}{\lambda}\right) d\mu\\
&\geq C\int_{\left\{y\in B_\delta:\Phi\left(\frac{f(y)}{\lambda}\right)>1\right\}} \Phi\left(\frac{f}{\lambda}\right) d\mu(y),
\end{aligned}$$

as we wished to prove. □

We also shall need the following lemma proved in [3], which is the corresponding result of the inequality (1.3) for $M_{\alpha,\Phi}$.

Lemma 2.5. [3] *Let (X, d, μ) be a space of homogeneous type, $0 \leq \alpha < 1$, Φ a Young function, $B = B(x, R)$ a fixed ball and $\tilde{B} = B(x, 2KR)$. Then, there exists a constant $C > 0$, depending only on the constants of the space, such that*

$$\max\left\{M_{\alpha,\Phi}(f\chi_{X\setminus\tilde{B}})(y), \mu(B)^\alpha M_\Phi(f\chi_{X\setminus\tilde{B}})(y)\right\} \leq C\inf_{z\in B} M_{\alpha,\Phi}(f\chi_{X\setminus\tilde{B}})(z),$$

for all $y \in B$.

3. Proof of Theorem 1.1

Without loss of generality we may assume that $\Phi(1) = 1$ and $\Psi(1) = 1$. To prove that Θ is a Young function we proceed as in [2]. In fact, let us assume that $\Phi(t) = \int_0^t \phi(u)\,du$. Since $\Phi(1) = 1$ we get

$$\Phi(t/u) = 1 + \frac{1}{u}\int_u^t \phi(v/u)\,dv, \quad \text{for } t \geq u.$$

Replacing this formula in (1.6) and changing the order of integration we get

$$\begin{aligned}
\Theta(t) &= \int_0^t \Psi_0'(u)\left[1 + \frac{1}{u}\int_u^t \phi(v/u)\,dv\right] du\\
&= \int_0^t \Psi_0'(u)\,du + \int_0^t \left[\int_0^v \Psi_0'(u)\phi(v/u)u^{-1}\,du\right] dv = \int_0^t \theta(u)\,du,
\end{aligned}$$

with

$$\theta(t) = \Psi_0'(t) + \int_0^t \Psi_0'(u)\phi(t/u)u^{-1}\, du.$$

It follows that Θ is a Young function, since Ψ_0' and ϕ are non-negative and Ψ_0' is increasing.

To prove (1.7), we begin proving that there exists $C > 0$ such that $M_{\alpha,\Theta}f(x) \leq CM_{\alpha,\Psi}(M_\Phi f)(x)$ for all $x \in X$. Let us assume that $f \geq 0$ and let us fix an $x \in X$ such that $M_{\alpha,\Psi}(M_\Phi f)(x) < \infty$. Let $B = B(z,R)$ any ball on X such that $x \in B$ and $\tilde{B} = B(z, 3K^2R)$. Notice that it is enough to show that there exists a constant C such that

$$||f||_{\Theta,B} \leq C||M_\Phi f||_{\Psi_0,\tilde{B}}. \tag{3.1}$$

Let δ be the quasi-distance equivalent to d defined in Theorem 2.3. If $B_\delta = B_\delta(z,R)$, let $\lambda_0 = ||M_\Phi f||_{\Psi_0,B_\delta}$. To prove (3.1) it is enough to show that there exists a constant $C_0 > 1$ such that

$$\frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) \leq 1. \tag{3.2}$$

In fact, from (2.6) we get that $B \subset B_\delta \subset \tilde{B}$. On the other hand, $\mu(\tilde{B}) \leq \tilde{C}\mu(B)$ for some universal constant $\tilde{C} \geq 1$. Since Θ is a convex function, if (3.2) holds then

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) &\leq \frac{\mu(\tilde{B})}{\mu(B)} \frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) \\
&\leq \frac{\tilde{C}}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) \\
&\leq \frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) \leq 1.
\end{aligned}$$

Thus,

$$\|f\|_{\Theta,B} \leq \tilde{C}C_0\lambda_0 = \tilde{C}C_0||M_\Phi f||_{\Psi_0,B_\delta} \leq \tilde{C}^2C_0||M_\Phi f||_{\Psi_0,\tilde{B}},$$

and we get inequality (3.1) with $C = \tilde{C}^2C_0$.

Now, by the definition of the function Θ we get that

$$\begin{aligned}
\int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) &= \int_{B_\delta} \int_0^{\frac{f(x)}{C_0\lambda_0}} \Psi_0'(u)\Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) du\, d\mu(x) \\
&= \int_1^\infty \Psi_0'(u) \int_{\left\{x\in B_\delta:\ \frac{f(x)}{C_0\lambda_0}>u\right\}} \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) d\mu(x)\, du \\
&\leq \int_1^\infty \Psi_0'(u) \int_{\left\{x\in B_\delta:\ \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right)>1\right\}} \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) d\mu(x)\, du.
\end{aligned}$$

Notice that in the last inequality we have used that the Young function Φ is strictly increasing for $t > 1$ (this is a consequence of the convexity and the assumption $\Phi(1) = 1$). Now, let us observe that, since $u > 1$ and $\Psi_0 \approx_\infty \Psi$, $C_0\lambda_0 u > C_0||M_\Phi f||_{\Psi_0,B_\delta} \geq$

$C_0C_1||M_\Phi f||_{\Psi,B_\delta} \geq C_0C_1||f||_{\Phi,\tilde{B}} \geq \frac{C_0C_1}{\tilde{C}}||f||_{\Phi,B_\delta}$, where $\tilde{C}$ is such that $\mu(\tilde{B}) \leq \tilde{C}\mu(B)$. Then, choosing C_0 such that $C_0C_1 \geq \tilde{C}$ and applying Lemma 2.4 we get that

$$\begin{aligned}\int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) &\leq C\int_1^\infty \Psi_0'(u)\mu(\{x \in B_\delta : M_\Phi f(x) > DC_0\lambda_0 u\})\, du \\ &\leq C\int_{B_\delta} \Psi_0\left(\frac{M_\Phi f(x)}{DC_0\lambda_0}\right) d\mu(x) \leq \frac{C\mu(B_\delta)}{DC_0}.\end{aligned}$$

Then, choosing $C_0 \geq \max\{CD^{-1}, \tilde{C}C_1^{-1}\}$, we clearly obtain (3.2).

Now, we shall prove the other inequality in (1.7), that is, there exists $C > 0$ such that $M_{\alpha,\Psi}(M_\Phi f)(x) \leq CM_{\alpha,\Theta} f(x)$ for all $x \in X$. Let $x \in X$ such that $M_{\alpha,\Theta} f(x) < \infty$. First, we shall show that there exists $C > 0$ such that

$$||M_\Phi f||_{\Psi_0,B} \leq C||f||_{\Theta,B}, \tag{3.3}$$

for any ball B such that $x \in B$ and for any function f with support in B. By an homogeneous argument, we may assume $||f||_{\Theta,B} = 1/2$, that is, $\frac{1}{\mu(B)}\int_B \Theta(2f(x))\, d\mu(x) \leq 1$. Now, applying (2.4) we get that

$$\begin{aligned}\int_B \Psi_0(M_\Phi f(x))\, d\mu(x) &= \int_0^\infty \Psi_0'(u)\mu(\{x \in B :\ M_\Phi f(x) > u\})\, du \\ &\leq C\int_1^\infty \Psi_0'(u)\left(\int_{\{x\in B:\ f(x)>u/2\}} \Phi\left(\frac{2f(x)}{u}\right) d\mu(x)\right) du \\ &= C\int_B \left(\int_1^{2f(x)} \Psi_0'(u)\Phi\left(\frac{2f(x)}{u}\right) du\right) d\mu(x) \\ &= C\int_B \Theta(2f(x))\, d\mu(x) \leq C\mu(B).\end{aligned}$$

So, we get (3.3) for any f with $\operatorname{supp}(f) \subset B$. For an arbitrary $f \geq 0$, let $x \in X$, $B = B(z,R)$ a ball such that $x \in B$ and $\tilde{B} = B(x, 2KR)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$. Then

$$\mu(B)^\alpha||M_\Phi f||_{\Psi_0,B} \leq \mu(\tilde{B})^\alpha||M_\Phi f_1||_{\Psi_0,\tilde{B}} + \mu(B)^\alpha||M_\Phi f_2||_{\Psi_0,B} = I + II.$$

By (3.3) we get that

$$I \leq C\mu(\tilde{B})^\alpha||f||_{\Theta,\tilde{B}} \leq M_{\alpha,\Theta} f(x).$$

To estimate II, let us observe that, as in [4], we can prove that $\Phi \prec_\infty \Theta$. In fact, notice that there exists $t_0 > 1$ such that $\Psi_0(u) \geq 1$ for all $u \geq t_0$, then for $t \geq 2t_0$,

$$\begin{aligned}\Theta(t) &= \int_1^t \Psi_0'(u)\Phi(t/u)\, du \\ &\geq \int_1^t \Psi_0(u)\Phi'(t/u)tu^{-2}\, du \\ &\geq \int_{t_0}^t \Phi'(t/u)tu^{-2}\, du\end{aligned}$$

$$= \int_1^{t/t_0} \Phi'(v)\,dv = c_0\Phi(t/t_0).$$

Then, $M_\Phi f(x) \le CM_\Theta f(x)$. Now, using twice Lemma 2.5 we get that

$$\begin{aligned} II &\le \mu(B)^\alpha || \inf_{z\in B} M_\Phi f_2(z)||_{\Psi_0,B} \\ &\le C\mu(B)^\alpha \inf_{z\in B} M_\Phi f_2(z) \\ &\le C\mu(B)^\alpha \inf_{z\in B} M_\Theta f_2(z) \le CM_{\alpha,\Theta} f(x). \end{aligned}$$

Finally, putting together the estimates for I and II we are done.

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