DIMENSION FUNCTIONS OF CANTOR SETS

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Abstract. We estimate the packing measure of Cantor sets associated to non-increasing sequences through their decay. This result, dual to one obtained by Besicovitch and Taylor, allows us to characterize the dimension functions recently found by Cabrelli *et al* for these sets.

1. Introduction

A Cantor set is a compact perfect and totally disconnected subset of the real line. In this article we consider Cantor sets of Lebesgue measure zero. Different kinds of these sets appear in many areas of mathematics, such as number theory and dynamical systems. They are also interesting in themselves as theoretical examples and counterexamples. A classical way to understand them quantitatively is through the Hausdorff measure and dimension.

A function $h : (0, \lambda_h] \to (0, \infty]$, where $\lambda_h > 0$, is said to be a *dimension function* if it is continuous, non-decreasing and $h(x) \to 0$ as $x \to 0$. We denote by $\mathscr{D}$ the set of dimension functions.

Given $E \subset \mathbb{R}$ and $h \in \mathscr{D}$, we set $h(E) = h(|E|)$, where $|E|$ is the el diameter of the set E.

Recall that a δ-covering of a given set E is a countable family of subsets of $\mathbb{R}$ covering E whose diameters are less than δ. The h-Hausdorff measure of E is defined as

$$\mathcal{H}^h(E) = \varliminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} h(U_i) : \{U_i\} \text{ is a } \delta \text{ covering of } E \right\}. \tag{1.1}$$

We say that E is a *h-set* if $0 < \mathcal{H}^h(E) < +\infty$.

When the dimension function is $g_s(x) = x^s$, for $s \geq 0$, we set $\mathcal{H}^s := \mathcal{H}^{g_s}$ ($\mathcal{H}^0$ is the counting measure). The Hausdorff dimension of the set E, denoted by $\dim E$, is the unique value t for which $\mathcal{H}^s(E) = 0$ if $s > t$ and $\mathcal{H}^s(E) = +\infty$ if $s < t$ (see Proposition 2.2). This property allows us to obtain an intuitive classification of how *thin* a subset of $\mathbb{R}$ of Lebesgue measure zero is.

A set E is said *dimensional* if there is at least one $h \in \mathscr{D}$ which makes E an h-set. Not all sets are dimensional (cf. [Bes39]), in fact, there are open problems about the dimensionality of certain sets, for instance, the set of Liouville numbers, which has Hausdorff dimension zero (see for example [Ols03]). Nevertheless, Cabrelli *et al* [CMMS04] showed that every Cantor set associated to a non-increasing sequence a is dimensional, that is, they constructed a function $h_a \in \mathscr{D}$ for which C_a is an h_a-set. Moreover, they show that if the sequence a behaves like $n^{-1/s}$, then $h_a \equiv g_s$

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(see definition below) and therefore C_a is an s-set. But in other cases the behavior of these functions is not so clear. For example, there exists a sequence a such that C_a is an α-set but $h_a \not\equiv g_\alpha$ ([CHM02]). So these functions could be too general in order to give a satisfactory idea about the size of the set. To understand this situation we study the packing premeasure of these sets, which is defined as follows. A *δ-packing* of a given set E is a disjoint family of open balls centered at E with diameters less than δ. The *h-packing premeasure* of E is defined as

$$P_0^h(E) = \overline{\lim_{\delta\to 0}} \left\{ \sum_{i=1}^{\infty} h(U_i) : \{B_i\}_i \text{ is a } \delta\text{-packing of } E \right\}.$$

As a consequence of our main result, which is dual to the one obtained in [BT54] and will be dealt with in Section 4, we are able to characterize completely when a dimension function is equivalent to a power function (Theorem 4.4). That is, for a non-increasing sequence a and $h \in \mathscr{D}$ we obtain that

$$h \equiv g_s \iff 0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty.$$

Thus, to have that $g_\alpha \equiv h_a$, it is not only necessary that C_a is an α-set but also that $P_0^\alpha(C_a) < +\infty$.

2. Some remarks and definitions

By the definition of P_0^h, it is clear that it is monotone but it is not a measure because it is not σ-additive; the *h-packing measure* $\mathcal{P}^h$ is obtained by a standard argument, $\mathcal{P}^h(E) = \inf\{ \sum_{i=1}^\infty P_0^h(E_i) : E = \bigcup_i E_i\}$.

As with Hausdorff measures, given a set E there exists a critical value $\dim_P E$, the *packing dimension* of E, such that $\mathcal{P}^s(E) = 0$ if $s > \dim_P E$ and $\mathcal{P}^s(E) = +\infty$ if $s < \dim_P E$. Analogously for the prepacking measure family $\{P_0^s\}$ we call ΔE its critical value. In [Tri82] it is shown that ΔE coincides with the upper Box dimension of E, which we now define.

Given $0 < \varepsilon < \infty$ and a non-empty bounded set $E \subset \mathbb{R}^d$, let $N(E, \varepsilon)$ be the smallest number of balls of radio ε needed to cover E. The *lower* and *upper Box dimensions* of E are given by

$$\underline{\dim}_B E = \underline{\lim_{\varepsilon\to 0}} \frac{\log N(E,\varepsilon)}{\log 1/\varepsilon} \qquad \text{and} \qquad \overline{\dim}_B E = \overline{\lim_{\varepsilon\to 0}} \frac{\log N(E,\varepsilon)}{\log 1/\varepsilon}$$

respectively. When the lower and upper limits coincide, the common value is the *Box dimension* of E and we denote it by $\dim_B E$.

Note that $\mathcal{H}^h(E) \leq \mathcal{P}^h(C_a)$ when h is a doubling function ([TT85]). Moreover, $\dim E \leq \dim_P E \leq \Delta E \leq 1$ and $\dim E \leq \underline{\dim}_B E \leq \overline{\dim}_B E$; all these inequalities can be strict ([Tri82]).

We observe that $\mathcal{H}^h$ is Borel-regular, which is also true if we only require the right continuity of $h \in \mathscr{D}$ [Rog98]. On the other hand, $\mathcal{P}^h$ and P_0^h are also Borel-regular, but in this case one has to require for $h \in \mathscr{D}$ to be left continuous ([TT85], Lemma 3.2). Since in this paper we are concerned with all of these measures, we require that $h \in \mathscr{D}$ should be continuous.

Now we define a partial order in $\mathscr{D}$, which will be our way of comparing the elements of $\mathscr{D}$.

Definition 2.1. Let f and h be in $\mathscr{D}$.

- f *is smaller than* h, denoted by $f \prec h$, if
$$\lim_{x\to 0} h(x)/f(x) = 0.$$
- f *is equivalent to* h, $f \equiv h$, if
$$0 < c_1 = \underline{\lim}_{x\to 0} \frac{h(x)}{f(x)} \le \overline{\lim}_{x\to 0} \frac{h(x)}{f(x)} = c_2 < +\infty.$$

We set $f \preceq g$ when $f \prec g$ or $f = g$. We say that f and g *are not comparable* if none of the relations $f \prec g$, $g \prec f$ or $f \equiv g$ holds.

Proposition 2.2. *If* ν^f *is* $\mathcal{H}^f$ *or* P_0^f *or* $\mathcal{P}^f$*, then we have the following*

i) If $f \prec h$ *then:* $\nu^f(E) < \infty \implies \nu^h(E) = 0$.

ii) If $f \equiv h$ *then:*

a) $\nu^f(E) < \infty \iff \nu^h(E) < \infty$.

b) $0 < \nu^f(E) \iff 0 < \nu^h(E)$.

c) In particular, E *is a* f*-set if and only if* E *is a* h*-set.*

The proof of this proposition for the Hausdorff measure case can be found in [Rog98]; the packing cases are analogous.

3. Cantor sets associated to non-increasing sequences

Let $a = \{a_k\}$ be a positive, non-increasing and summable sequence. Let I_a be a closed interval of length $\sum_{k=1}^{\infty} a_k$. Denote by $\mathscr{C}_a$ the family of all closed sets E contained in I_a which are of the form $E = I_a \setminus \bigcup_{j\ge 1} U_j$, where $\{U_j\}$ is a disjoint family of open intervals contained in I_a such that $|U_k| = a_k$ $\forall k$. Thus, every element of $\mathscr{C}_a$ has Lebesgue measure zero.

From this family, we consider the Cantor set C_a associated to the sequence a constructed as follows: In the first step, we remove from I_a an open interval of length a_1, resulting two closed intervals I_1^1 and I_2^1. Having constructed the k-th step, we get the closed intervals $I_1^k, \ldots, I_{2^k}^k$ contained in I_a. The next step consists in removing from I_j^k an open interval of length a_{2^k+j}, obtaining the closed intervals I_{2j-1}^{k+1} and I_{2j}^{k+1}. Then we define $C_a := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^k$. Note that in this construction there is a unique form of removing open intervals at each step; also, note that for this construction it is not necessary for the sequence to be non-increasing.

Remark 3.1. Since a is non-increasing, the sequence $\{|I_j^k|\}_{(k,j)}$, with $1 \le j \le 2^n$ and $k \ge 1$, is (lexicographically) non-increasing.

We associate to the sequence a the summable sequences $\underline{a}$ and $\overline{a}$ defined as
$$\underline{a}_n = a_{2^r-1}, \text{ with } 2^{r-1} \le n < 2^r \text{ and}$$
$$\overline{a}_n = a_{2^r}, \text{ with } 2^r \le n < 2^{r+1};$$
thus $\underline{a}_n \le a_n \le \overline{a}_n$ $\forall n$. The sets $C_{\underline{a}}$ y $C_{\overline{a}}$ are *uniform*, which means that for each $k \ge 1$, the closed intervals of the k-th step have all the same length. Observe that for each uniform Cantor set the Hausdorff and lower box dimensions coincide (cf. [CHM97]).

To study the Hausdorff measure and dimension of these sets, Besicovitch and Taylor in [BT54] studied the decay of the sequence $b_n = r_n/n$, where $r_n = \sum_{j\ge n} a_j$. They introduced the number
$$\alpha(a) = \underline{\lim}_{n\to\infty} \alpha_n, \text{ where } n b_n^{\alpha_n} = 1 \ \forall n, \tag{3.1}$$

and showed that $\dim E \leq \alpha(a)$ for all $E \in \mathscr{C}_a$. In fact, $\dim C_a = \alpha(a)$ (see for example [CMMS04]). Also, as a consequence of another result in [BT54] (see Proposition 4.1 below),

$$\dim C_a = \inf\{s > 0 : \underline{\lim}\, nb_n^s < +\infty\}. \tag{3.2}$$

In this paper (Theorem 4.2) we obtain the symmetrical result

$$\overline{\dim}_B C_a = \Delta C_a = \inf\{s > 0 : \overline{\lim}\, nb_n^s < +\infty\}.$$

On the other hand, the box dimensions of the sets in $\mathcal{C}_a$ are related to the constants

$$\gamma(a) = \underline{\lim}_{k\to\infty} \frac{\log 1/k}{\log a_k} \quad \text{y} \quad \beta(a) = \overline{\lim}_{k\to\infty} \frac{\log 1/k}{\log a_k}. \tag{3.3}$$

In fact, by Propositions 3.6 and 3.7 of Falconer's book [Fal97], the box dimension of $E \in \mathscr{C}_a$ exists if and only if $\gamma(a) = \beta(a)$, and in this case $\dim_B E = \beta(a)$. Moreover, every $E \in \mathscr{C}_a$ has upper box dimension $\beta(a)$, which in [Tri95] is shown to be equal to $\overline{\lim}_{n\to\infty} \alpha_n$. We do not know if every set in $\mathscr{C}_a$ has the same lower box dimension, but as the following proposition shows, for C_a there is a symmetry between $\underline{\lim}\, \alpha_n$ and $\overline{\lim}\, \alpha_n$ with respect to the box dimensions (Here we use the non-increasingness of a).

Proposition 3.2. *If a is a non-increasing sequence then $\underline{\dim}_B C_a = \dim C_a = \alpha(a)$.*

Proof. First note that $\underline{\lim}\, nb_n^s \sim \underline{\lim}\, n\overline{b}_n^s$. In fact, to see the nontrivial inequality, if $\{n_j\}$ is a subsequence of the natural numbers let $2^{l_j - 1} \leq n_j < 2^{l_j}$, so $2^{l_j} b_{2^{l_j}}^s \leq 2n_j b_{n_j}^s$ and therefore $\underline{\lim}_{k\to\infty} 2^k b_{2^k}^s \leq 2\, \underline{\lim}_{n\to\infty} nb_n^s$. Now for $2^{j_n} \leq n < 2^{j_n+1}$,

$$\begin{aligned} \overline{r}_n &\leq \sum_{k \geq 2^{j_n}} \overline{a}_k = \sum_{k=0}^{\infty} 2^{j_n+k} a_{2^{j_n+k}} \\ &\leq 2 \sum_{k=0}^{\infty} 2^{j_n+k-1} a_{2^{j_n+k}-1} = 2\ \underline{r}_{2^{j_n-1}} \leq 2\ r_{2^{j_n-1}}, \end{aligned}$$

hence $\underline{\lim}_{n\to\infty} n\overline{b}_n^s \leq 4\, \underline{\lim}_{j\to\infty} 2^j b_{2^j}^s \leq 8\, \underline{\lim}_{n\to\infty} nb_n^s$.

Then by (3.2) $\dim C_a = \dim C_{\overline{a}}$, and by Proposition 3.1 of [CHM97] we have that $\dim C_a = \underline{\dim}_B C_{\overline{a}}$. On the other hand, $N(\varepsilon, C_a) \leq N(\varepsilon, C_{\overline{a}})$ (C_a can be mapped to $C_{\overline{a}}$ by a bijection which preserves the order; then, if two points of $C_{\overline{a}}$ are contained in an open set U, the corresponding points of C_a will be contained in an open set of the same diameter as U), so $\underline{\dim}_B C_a \leq \underline{\dim}_B C_{\overline{a}}$. □

4. Main results

The next proposition is a result that generalizes the one established in [BT54] for the functions g_s, to any function $h \in \mathscr{D}$ (c.f.[CHM02]). It shows that $\mathcal{H}^h(C_a)$ behaves like $nh(b_n)$ when $n \to \infty$.

Proposition 4.1. *For $h \in \mathscr{D}$, $\frac{1}{4}\, \underline{\lim}_{n\to\infty} nh(b_n) \leq \mathcal{H}^h(C_a) \leq 4\, \underline{\lim}_{n\to\infty} nh(b_n)$.*

Proof. The lower inequality $\frac{1}{4}\, \underline{\lim}_{n\to\infty} nh(b_n) \leq \mathcal{H}^h(C_a)$ is obtained in exactly the same way than in [BT54] replacing g_s by h. Hence, we only show here that

$\mathcal{H}^h(C_a) \leq 4 \varliminf nh(b_n)$. We begin by noting that $|\underline{I}_j^k| = \sum_{i=0}^{\infty} 2^i \underline{a}_{k+1+i} = \underline{b}_{2^k}$ for $1 \leq j \leq 2^k$, $k > 0$. Then, from the identities

$$|I_1^k| = a_{2^k} + (a_{2^{k+1}} + a_{2^{k+1}+1}) + (a_{2^{k+2}} + a_{2^{k+2}+1} + a_{2^{k+2}+2} + a_{2^{k+2}+3}) + \dots\ ,$$
$$|\underline{I}_1^{k-1}| = a_{2^k-1} + (a_{2^{k+1}-1} + a_{2^{k+1}-1}) + (a_{2^{k+2}-1} + a_{2^{k+2}-1} + a_{2^{k+2}-1} + a_{2^{k+2}-1}) + \dots$$

and Remark 3.1, we have the following estimate

$$|I_j^k| \leq |I_1^k| \leq |\underline{I}_1^{k-1}| = \underline{b}_{2^{k-1}} \leq b_{2^{k-1}}, \tag{4.1}$$

and hence $\sum_{j=1}^{2^k} h(I_j^k) \leq 2^k h(b_{2^{k-1}})$. Given $\delta > 0$ there exists k_δ such that $|I_1^k| < \delta$ for $k \geq k_\delta$, that is, for $k \geq k_\delta$, the closed intervals of the k-th step form a δ-covering of C_a which implies $\mathcal{H}^h(C_a) \leq 2 \varliminf_{k\to\infty} 2^k h(b_{2^k})$, and therefore $\mathcal{H}^h(C_a) \leq 4 \varliminf nh(b_n)$. □

Our main result is the following theorem, which is in some sense dual to the previous one.

Theorem 4.2. *For any $h \in \mathscr{D}$ $\frac{1}{8} \varlimsup_{n\to\infty} nh(b_n) \leq P_0^h(C_a) \leq 8 \varlimsup_{n\to\infty} nh(b_n)$.*

Proof. For the first inequality, suppose that $\varlimsup_{n\to\infty} nh(b_n) > d$. To prove that $P_0^h(C_a) \geq d/8$, for each $\delta > 0$ it suffices to find a δ-packing $\{B_i\}_i$ of C_a with $\sum_i h(B_i) > d/8$. Observe that, since $\{a_n\}$ is non-increasing, then

$$h(b_{2^k}) = h\left(2^{-k} \sum_{1\leq i\leq 2^k} |I_i^k|\right) \leq h\left(I_1^k\right). \tag{4.2}$$

By hypothesis there exists a subsequence $\{n_j\}_{j\geq 1}$ such that $n_j h(b_{n_j}) > d$. For each j, let k_j be the unique integer for which $2^{k_j} \leq n_j < 2^{k_j+1}$; since $\{b_n\}_n$ is decreasing, it follows from (4.2) that

$$d < n_j h(b_{n_j}) < 2^{k_j+1} h(b_{2^{k_j}}) \leq 2^{k_j+1} h\left(I_1^{k_j}\right). \tag{4.3}$$

Pick j big enough so that $|I_1^{k_j}| < \delta$; since the diameter of this interval is smaller than the diameter of every interval of the $k_j - 1$ step, the family of intervals $\{B_i\}_{i=1}^{2^{k_j-2}}$, where B_i is centered at the right endpoint of the interval $I_{2i-1}^{k_j-1}$ and $|B_i| = |I_1^{k_j}|$, turns out to be a δ-packing of C_a, and by (4.3), $\sum_i h(B_i) = 2^{k_j-2} h(I_1^{k_j}) > d/8$.

For the second inequality, if $\{B_i\}_{i=1}^N$ is a δ-packing of C_a, we define

$$k_i = \min\{k : I_j^k \subset B_i \text{ for some } 1 \leq j \leq 2^k\}.$$

By the definition of k_i, B_i is centered at a point of an interval of the $k_i - 1$ step but it does not contain the interval, so $|B_i| < |I_{j_i}^{k_i-2}|$, where $I_{j_i}^{k_i-2}$ is the interval of the $k_i - 2$ step which contains the center of B_i. Then, by the monotony of h and (4.1),

$$\sum_{i=1}^N h(B_i) \leq \sum_{i=1}^N h(I_{j_i}^{k_i-2}) \leq \sum_{i=1}^N h(b_{2^{k_i-3}}). \tag{4.4}$$

Further, we can assume that $|B_1| \geq \ldots \geq |B_N|$, so $k_1 \geq \ldots \geq k_N$. Let $l_1 > \ldots > l_M$ denote those k_i's that do not repeat themselves. Let θ_m be the number of repeated l_m's, i.e., θ_m tells us how many of the B_i's contain an interval of the l_m-th step but none of the previous ones. Since $\{B_i\}_{i=1}^N$ is a disjoint family, θ_1 cannot be greater than the number of intervals of step l_1, which is 2^{l_1}; each ball of the packing associated to l_1 contains $2^{l_2-l_1}$ intervals of step l_2 and therefore $\theta_2 \leq 2^{l_2} - \theta_1 2^{l_2-l_1}$. Continuing with this process we obtain

$$\theta_M \leq 2^{l_M} - \sum_{i=1}^{M-1} \theta_i 2^{l_M-l_i} = 2^{l_M}\left(1 - \sum_{i=1}^{M-1} \frac{\theta_i}{2^{l_i}}\right),$$

and hence $\sum_{i=1}^{M} \theta_i/2^{l_i} \leq 1$.

Finally, choose δ sufficiently small such that $2^{l_1-3} \geq n_0$, where

$$\sup_{n\geq n_0} nh(b_n) \leq \overline{\lim_{n\to\infty}}\, nh(b_n) + \varepsilon.$$

Then,

$$\sum_{i=1}^{N} h(B_i) \leq \sum_{j=1}^{M} \frac{\theta_j}{2^{l_j-3}} 2^{l_j-3} h(b_{2^{l_j-3}}) \leq 8(\overline{\lim_{n\to\infty}}\, nh(b_n) + \varepsilon)\ ,$$

and from this the theorem follows. □

We are now ready to complete the characterization promised in the introduction.

Note that the function $h_a \in \mathscr{D}$ found in [CMMS04] is defined in such a way that $h_a(b_n) = 1/n$ for all n. Then, by Theorem 4.2 we obtain that $P_0^{h_a}(C_a) < +\infty$, and therefore the Cantor sets associated to non-increasing sequences not only are dimensional but also have a dimension function which simultaneously *regularizes* the covering and packing processes in the construction of the measures. We need the following Lemma.

Lemma 4.3. *Let $h, g \in \mathscr{D}$. In addition assume that $0 < \mathcal{H}^h(C_a) \leq P_0^h(C_a) < +\infty$. Then*

a) $h \equiv g \iff 0 < \underline{\lim}_{n\to\infty} ng(b_n) \leq \overline{\lim}_{n\to\infty} ng(b_n) < +\infty$;
b) $g \prec h \iff \lim_{n\to\infty} ng(b_n) = +\infty$;
c) $h \prec g \iff \lim_{n\to\infty} ng(b_n) = 0$.

Proof. By Proposition 4.1 and Theorem 4.2, since $\mathcal{H}^h(C_a) > 0$ and $P_0^h(C_a) < +\infty$, there are costants $0 < c_h$ and $C_h < +\infty$ such that $c_h 1/n \leq h(b_n) \leq C_h 1/n$, and all three necessary conditions follow. On the other hand, if $\{y_j\}$ is a sequence which decreases to 0, then there exists a subsequence $\{n_j\}$ such that $b_{n_j+1} \leq y_j < b_{n_j}$. Therefore

$$\frac{g(y_j)}{h(y_j)} \leq \frac{g(b_{n_j})}{h(b_{n_j+1})} \leq 2c_h^{-1} n_j g(b_{n_j})$$

and

$$\frac{g(y_j)}{h(y_j)} \geq (2C_h)^{-1}(n_j+1) g(b_{n_j+1});$$

from which the sufficient conditions also follow. □

We have now the main theorem of this part.

Theorem 4.4. *Let a be a non-increasing sequence, and let $h \in \mathscr{D}$ be such that $0 < \mathcal{H}^h(C_a) \leq P^h(C_a) < +\infty$. Then, for $g \in \mathscr{D}$ we have:*

a) $h \equiv g \iff 0 < \mathcal{H}^g(C_a) \leq P_0^g(C_a) < +\infty$;
b) $g \prec h \iff \mathcal{H}^g(C_a) = +\infty$;
c) $h \prec g \iff P_0^g(C_a) = 0$.

In particular, h will be equivalent to x^s if and only if $0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty$.

Proof. The proof is inmediate from Proposition 4.1, Theorem 4.2 and Lemma 4.3. □

Corollary 4.5. *Let $\gamma = \gamma(a)$, $\beta = \beta(a)$ and $\alpha = \dim C_a$. Take h_a to be the dimension function of C_a.*

a) If $s < \alpha$ and $\beta < t$ then $g_s \prec h_a$ and $h_a \prec g_t$. If $\mathcal{H}^\alpha(C_a) = +\infty$ then $g_\alpha \prec h_a$, and if $\mathcal{H}^\alpha(C_a) < +\infty$ then $g_\alpha \not\prec h_a$. If $P_0^\beta(C_a) = 0$ then $h_a \prec g_\beta$, and if $P_0^\beta(C_a) > 0$ then $h_a \not\prec g_\beta$.

b) In the case $\gamma < \beta$, $h_a \not\equiv g_s$ for no $s \geq 0$. Moreover, if $\alpha < t < \beta$ then h_a and g_t are not comparable. In the limit cases, if $\mathcal{H}^\alpha(C_a) < +\infty$ then h_a and g_α are not comparable, and if $P_0^\beta(C_a) > 0$, h_a and g_β are not comparable.

Proof. In the case $\mathcal{H}^\alpha(C_a) < +\infty$, if g_a were to satisfy $g_\alpha \prec h_a$, Proposition 2.2 implies that $\mathcal{H}^{h_a}(C_a) = 0$, which is a contradiction. Hence $g_\alpha \not\prec h_a$. Analogously, $P_0^\beta(C_a) > 0$ implies that $h_a \not\prec g_\beta$, for if not $P_0^{h_a}(C_a) = +\infty$, contradicting Theorem 4.2. The rest of the claims of item *a*) are immediate from Theorem 4.4.

To show *b*), if $h_a \equiv g_s$ for some $s \geq 0$ then $0 < \mathcal{H}^s(C_a) \leq P_0^s(C_a) < +\infty$, therefore $\underline{\dim}_B C_a = \overline{\dim}_B C_a$ and hence $\gamma = \beta$. Moreover, by Proposition 2.2, it follows that $g_s \not\prec h$ when $s > \dim C_a$, and also $h_a \not\prec g_s$ if $0 \leq s < \beta$. □

Note that $\gamma < \beta$ if and only if $\dim C_a < \overline{\dim}_B C_a$, so part *b*) of this corollary emphasizes that in order to have $h_a \equiv g_s$ we need $\gamma = \beta$. But this latter condition and the fact that C_a is an α-set are not sufficient to ensure the equivalence, and thus the hypothesis of Theorem 4.4 *a*) cannot be weakened to *existence of box dimension and C_a being an α-set.*

Example 4.6. *For each $0 < s < 1$ there exists a Cantor set C_a associated to a nonincreasing sequence for which $\dim C_a = \overline{\dim}_B C_a = s$ and $0 < \mathcal{H}^s(C_a) < +\infty$, but $P_0^s(C_a) = +\infty$.*

To construct this sequence we set $\lambda_k = \left(\frac{1}{2}\right)^{\frac{k}{s+\varepsilon_k}}$, where

$$\varepsilon_k = \begin{cases} \frac{s \log l}{k}, & 2^m < k \leq 2^{m+1}, l = k - 2^m,\ m \text{ even} \\ 0, & \text{otherwise} \end{cases}.$$

Let us define $a_j = \lambda_k$ for $2^{k-1} \leq j < 2^k$ and $k \geq 1$. The uniform Cantor set C_a gives us the example. In fact, it is easy to check that a is summable, decreasing and, using (3.3), that $\dim C_a = \overline{\dim}_B C_a = s$. Next we check the claims about the measures:

a) $0 < \underline{\lim}_{n\to\infty} n b_n^s$: Since $b_{2^k} = \sum_{i\geq 0} 2^i \lambda_{k+1+i}$ and $\lambda_j \geq (\frac{1}{2})^{\frac{j}{s}}$ $\forall j$, we have that

$$2^k b_{2^k}^s \geq 2^k \left(\sum_{i\geq 0} \left(\frac{1}{2}\right)^{\frac{k+1+i}{s} - i} \right)^s = \frac{1}{2} \left(\sum_{i\geq 0} \left(\frac{1}{2}\right)^{(\frac{1}{s}-1)i} \right)^s = c_s > 0$$

and remember that $\underline{\lim}_{n\to\infty} n b_n^s \sim \underline{\lim}_{k\to\infty} 2^k b_{2^k}^s$.

b) $\overline{\lim}_{n\to\infty} n b_n^s = +\infty$: Let $n_j = 2^{2^j - 1}$, j odd. Then $b_{n_j} = \sum_{i\geq 0} 2^i \lambda_{2^j+i} > \lambda_{2^j}$, therefore

$$n_j b_{n_j}^s \geq 2^{2^j-1} \lambda_{2^j}^s = 2^{\frac{2^j (j-1)\log 2}{2^j + (j-1)\log 2} - 1},$$

which increases to infinity with j.

c) $\underline{\lim}_{n\to\infty} nb_n^s < +\infty$: Now we set $n_j = 2^{2^j}$ for j odd and observe that

$$n_j b_{n_j}^s = n_j \left(\sum_{i\geq 0} 2^i \lambda_{2^j+1+i}\right)^s = \left(\sum_{i\geq j}\sum_{l=1}^{2^i} 2^{2^i+l+2^j(\frac{1-s}{s})-1}\lambda_{2^i+l}\right)^s . \tag{4.5}$$

Each sum in l is bounded by a geometric term, more precisely, if j is sufficiently large there is a constant C_s depending only on s such that

$$\sum_{l=1}^{2^i} 2^{2^i+l+2^j(\frac{1-s}{s})}\lambda_{2^i+l} \leq C_s \left(\frac{1}{2}\right)^{i-j},$$

or equivalently,

$$\sum_{l=1}^{2^i} 2^l \lambda_{2^i+l} \leq C_s \left(\frac{1}{2}\right)^{2^i+2^j(\frac{1-s}{s})+i-j}. \tag{4.6}$$

This is easy to check when i is odd. For i even we obtain $\sum_{l=1}^{2^i} 2^l\lambda_{2^i+l} < C_s\left(\frac{1}{2}\right)^{\frac{2^i}{s+\varepsilon}}$ for small ε. Thus (4.6) will hold if

$$2^i\left(\frac{1-(s+\varepsilon)}{s+\varepsilon}\right) \geq 2^j\left(\frac{1-s}{s}\right) + i - j. \tag{4.7}$$

But notice that (4.7) is true for all $i \geq j$ choosing ε sufficiently small and j large enough so that $\varepsilon_{2^j} < \varepsilon$.

Remark 4.7. Proposition 4.1 and Theorem 4.2 are not valid in general as the following arguments show.

If a is a non-increasing sequence and $\tilde{a}$ is any rearrangement of a then $r_n^a \leq r_n^{\tilde{a}}$; hence, by (3.2) and since $C_{\tilde{a}} \in \mathscr{C}_a$,

$$\inf\{s>0 : \underline{\lim}\, n(b_n^{\tilde{a}})^s < +\infty\} \geq \dim C_a \geq \dim C_{\tilde{a}},$$

and each positive and summable sequence a has a rearrangement z for which $\dim C_z = 0$ (cf. [CMPS05]).

In the case of Theorem 4.2, let $\eta(a) := \inf\{s>0 : \overline{\lim}\, nb_n^s < +\infty\}$ and consider $\beta(a)$ as defined in Section 3. Note that if $t > \eta(a)$ then $a_n < Cn^{1-1/t}\ \forall n$, and hence $\beta(a) \leq \frac{t}{1-t}$ which implies that $\beta(a) \leq \frac{\eta(a)}{1-\eta(a)}$. Therefore $\frac{\beta(a)}{1+\beta(a)} \leq \eta(a)$. Thus, to show that this proposition fails in general, we exhibit a non-increasing sequence a and a rearrangement $\tilde{a}$ of it for which

$$\beta(a) < \frac{\beta(\tilde{a})}{1+\beta(\tilde{a})} \tag{4.8}$$

and therefore $\overline{\dim}_B C_{(\tilde{a})} = \beta(a) < \eta(\tilde{a})$. For $a_n = \left(\frac{1}{n}\right)^p$, we set

$$\tilde{a}_n = \begin{cases} a_{3^k}, & n = \lceil \log 3^k \rceil,\ n \neq 3^j\ \forall j \\ a_{\lceil \log 3^k \rceil}, & n = 3^k,\ n \neq \lceil \log 3^j \rceil\ \forall j \\ a_n, & \text{otherwise} \end{cases},$$

where $\lceil s \rceil$ denotes the smallest integer greater than s. Observe that this is a rearrangement of a since the sequence $\{\lceil \log 3^k \rceil\}$ is strictly increasing and $\lceil \log 3^{3^l} \rceil \neq 3^j$ for each l and each j. Then it is easy to check that $\beta(a) = 1/p$ and that $\beta(\tilde{a}) = \frac{\log 3}{p}$. Therefore (4.8) holds for $p > \frac{\log 3}{\log 3 - 1}$.

However these propositions keep on being true if we ask the Cantor set to be uniform with no further assumptions on the sequence. This can be seen in Theorem 4.2 since inequalities (4.3) and (4.4), where the decay assumption is needed, are still true for uniform Cantor sets. For Proposition 4.1 note that Lemma 4 of [BT54] does not need the decay. Moreover, to each Cantor set associated to a non-increasing sequence corresponds an uniform Cantor set with equivalent h-Hausdorff measure and h-packing premeasure.

Finally we give an application of the results of this section.

Let $p > 1$ and $q \in \mathbb{R}$. Consider the sequence a defined by $a_n = (\log n)^q/n^p$, $\forall\, n > 1$. We denote by $C_{p,q}$ the Cantor set associated to a and define the dimension function $h_{p,q}(x) = x^{\frac{1}{p}}/(-\log x)^{\frac{q}{p}}$.

Since $\gamma(a) = \beta(a) = 1/p$, we have that $\dim C_{p,q} = \dim_B C_{p,q} = 1/p$ for any q. Moreover, if $q = 0$, $C_p = C_{p,0}$ is an $1/p$-set (cf. [CMPS05]). Even more, as in this case $r_n \sim 1/n^{p-1}$, Theorem 4.2 implies that $P_0^{\frac{1}{p}}(C_a) < +\infty$. The next corollary extends these results.

Corollary 4.8. *With the above notation,* $0 < \mathcal{H}^{h_{p,q}}(C_{p,q}) \leq P_0^{h_{p,q}}(C_{p,q}) < +\infty$. *In particular,* $C_{p,q}$ *is an* $h_{p,q}$*-set.*

Proof. We show that $r_n \sim (\log n)^q/n^{p-1}$ and from this it is easy to see that $h_{p,q}(b_n) \sim 1/n$. First suppose that $q \geq 0$. In this case we have that

$$r_n \geq (\log n)^q \sum_{k\geq n} k^{-p} \geq c_1 \frac{(\log n)^q}{n^{p-1}}$$

and

$$r_n \leq \int_{n-1}^{\infty} \frac{(\log t)^q}{t^p} dt,$$

so integrating by parts we get

$$\begin{aligned} r_n &\leq \frac{1}{p-1}\left(\frac{(\log(n-1))^q}{(n-1)^{p-1}} + q\int_{n-1}^{\infty}\frac{(\log t)^{q-1}}{t^p}dt\right) \\ &= c_p \frac{(\log n)^q}{n^{p-1}} + O\left(\frac{(\log n)^{q-1}}{n^{p-1}}\right) \\ &\leq c_2 \frac{(\log n)^q}{n^{p-1}}, \end{aligned}$$

where c_1 and c_2 are constants depending only of p and q.

Now suppose that $q < 0$. In this case is easy to see that $r_n \leq c_3(\log n)^q/n^{p-1}$. On the other hand, integrating by parts twice and since $q(q-1) > 0$ we obtain

$$\begin{aligned} r_n &\geq \frac{1}{p-1}\left(\frac{(\log n)^q}{n^{p-1}} + \frac{q}{p-1}\frac{(\log n)^{q-1}}{n^{p-1}} + \frac{q(q-1)}{p-1}\int_n^{\infty}\frac{(\log t)^{q-2}}{t^{p-1}}\,dt\right) \\ &\geq \frac{1}{p-1}\frac{(\log n)^q}{n^{p-1}}\left(1 + \frac{q}{p-1}(\log n)^{-1}\right); \end{aligned}$$

taking n sufficiently large it follows that $r_n \geq c_4(\log n)^q/n^{p-1}$. □

Note that $h_{p,q} \preceq h_{s,t}$ if and only if $(1/p, q) \leq_l (1/s, t)$ ($\leq_l$ denote the lexicographical order in $(0,1)\times\mathbb{R}$). Define $\mathcal{H}^{\frac{1}{p},q} = \mathcal{H}^{h_{p,q}}$, so that $\mathcal{H}^{\frac{1}{p},0} = \mathcal{H}^{\frac{1}{p}}$. As a consequence of the above result we can conclude the following. If $q < 0$, $(1/p, q) <_l (1/p, 0)$, and since $\mathcal{H}^{p,q}(C_{p,q}) < +\infty$, we have that $\mathcal{H}^{\frac{1}{p}}(C_{p,q}) = 0$. On the other hand, if $q > 0$, then $(1/p, 0) <_l (1/p, q)$ and $\mathcal{H}^{p,q}(C_{p,q}) > 0$ implies that $\mathcal{H}^{\frac{1}{p}}(C_{p,q}) = +\infty$. Hence, the family $\{h_{p,q}\}$ provides a more accurate classification than the usual $\{g_s\}$, i.e., it distinguishes more sets; further, the dimension induced by this family can be seen as the set $((0,1)\times\mathbb{R})\cup\{(0,0)\}\cup\{(1,0)\}$ ordered with $\leq_l$, and the set function dim turns out to be the restriction of this order to $(0,1)\times\{0\}$.

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