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AN EFFICIENT TECHNIQUE FOR OPTIMAL TRUSS DESIGN

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Abstract.
A new algorithm is presented for size optimization of truss structures considering any kind of smooth objectives and constraints, together with constraints on the collapse loading obtained by limit analysis for loading conditions. The main difficulty of this problem is the fact that the collapse loading is a nonsmooth function on the design variables. In this paper we avoid nonsmooth optimization techniques. Our approach is based on a Feasible Directions Interior Point Algorithm for nonlinear constrained optimization. Three illustrative examples are discussed. The numerical results show that the calculation effort when limit analysis constraints are included is only slightly increased with respect to classic constraints.

1 INTRODUCTION

This work presents a new method for the automatic optimal design of truss structures, to achieve the minimum of a given objective function. The design variables represent the cross-sectional areas of bars or groups of bars. The truss may be subject to multiple load cases. The structural geometry is given and remains fixed (unchanged).

Previous works dealing with the optimal design of truss structures considering elastic constraints were published by Kirsch(1981)[1], Schmidt and Miura(1976)[2], Allwood and Chang(1984)[3], Ohasaki(2001)[4], Gil and Andreu (2001)[5], among others. Literature concerning optimal design involving plastic constraints is limited, however we can still mention Lee and Gordon(1981)[6] and Kirsch(1981)[1]. Publications concerning simultaneous constraints of elasticity and plasticity in the optimal truss design are rare. The work by Rohan and Whiteman(2000)[7] is also interesting because it is applied to both trusses and continua.

For statically determinate structures, the first yield condition is the ultimate capacity of the structure. However, for structures with multiple degrees of indeterminacy, the collapse loading is normally much higher than the loading that produces the first yield condition. The ultimate capacity of a structure has become an important issue in truss structural design. Limit analysis is an alternative analytical procedure for obtaining the ultimate loading of a structure for collapse. It determines the maximum safety factor, or factor of loading amplification, that can be supported by a structure of ideal elastoplastic materials subjected to a specified external loading. In optimal design, constraints on the safety factors with respect to the plastic collapse can then be introduced.

In this paper we consider the optimal design of structures with simultaneous constraints on smooth functions, like the elements stresses or nodal displacement, and on the safety factor in relation to the plastic collapse. The main difficulty of this problem is the fact that the critical load is a nonsmooth function of the design variables.

The present technique substitutes the limit analysis constraints by a set of linear constraints and the resulting problem is solved with an Interior Point Algorithm for smooth nonlinear optimization. Some test problems are solved in a very efficient way.

In the next section the Limit Analysis technique is first described. Then, the sensitivity of the critical load to changes in the structure is discussed. The optimization problem is presented in Section 5 and the numerical optimization procedure is presented in Section 6. Some numerical results are presented in the following section and finally we make our concluding remarks.

2 ABOUT LIMIT ANALYSIS

Limit analysis is the determination of the maximum load factor amplification that can be supported by a structure of ideal elasto-plastic material submitted to specified external loadings, Feijóo and Zouain (1988, 1989)[8,9], Zouain et al. (1993)[10]. We consider a Limit Analysis formulation in terms of static forces based on the Static Limit Analysis theorem that leads to a Linear Programming problem, Christiansen(1981)[11] and Zouain et al. (1993)[10]. Let us assume statical proportional loading where $\bar{\alpha}$ is the collapse amplification factor. That is, $\bar{\alpha}$ is the maximum of the amplification factors $\alpha \in \Re$ for which there is a plastically feasible stress distribution in equilibrium with α times the external loading vector. Let be ne the number of elements, ndf the number of degrees of freedom of the structure. Hence,

$$\bar{\alpha} = \begin{cases} \quad \max\, \alpha \\ \quad\; \alpha, T \\ \text{such that:} \quad B^T T - \alpha P = 0 \\ \qquad\qquad\qquad T - R \leq 0 \end{cases} \tag{1}$$

where $T \in \Re^{ne}$ is the vector of internal forces, $P \in \Re^{ndf}$ is the loading vector and $B \in \Re^{ndf \times ne}$ is the global deformation matrix. The independent variables of the Linear Program are α and T. The computation of B is obtained by assembling the constant matrix of deformation of each bar which is carried out in a similar way as in Zouain el al.(1993).

The inequality constraints represent the condition of plastic feasibility, where $R \in \Re^{ne}$ is a linear function of the bars cross section. We assume for the sake of simplicity that the yield stresses are the same for traction and compression.

Since in this paper, the design variables $x \in \Re^n$ are the cross -sectional areas of the bars or groups of bars, we have that R also depends linearly of x and B and P are constant. We can write $R = MLx$, where $M \in \Re^{ne \times ne}$ is a diagonal matrix with yield stresses and $L \in \Re^{ne \times n}$ is a boolean matrix relating the bars with the design variables.

Introducing now the design variables in the Limit Analysis, we have the collapse amplification factor as a function of x:

$$\bar{\alpha}(x) = \begin{cases} \quad \max\, \alpha \\ \quad\; \alpha, T \\ \text{such that:} \quad B^T T - \alpha P = 0 \\ \qquad\qquad\qquad T - MLx \leq 0 \end{cases} \tag{2}$$

3 SENSITIVITY STUDY OF LIMIT ANALYSIS

We discuss now the sensitivity of $\bar{\alpha}(x)$ with respect to the design variables x. According to the sensitivity theorem for Linear Programming described by Sobieski et al(1982)[12], Luenberger(1984)[13] and Haftka(1985)[14], we have:

$$\frac{\partial \bar{\alpha}(x)}{\partial R} = \lambda \tag{3}$$

Then,

$$\nabla \bar{\alpha}(x) = \frac{\partial \bar{\alpha}(x)}{\partial R} \frac{\partial R}{\partial x} = [M\, L]^T \lambda \tag{4}$$

where $\lambda \in \Re^{ne}$ are the Lagrange Multipliers corresponding to the inequality constraints of Problem (1). However, this expression is only valid if the set of active inequality constraints of the Linear Program (2) remains unchanged in a neighborhood of x. Then, $\bar{\alpha}(x)$ is nonsmooth, since the gradient does not exist at points x where the active set changes. As the derivatives of all the constraints of Problem (1) are constant with respect to x, it is easy to prove that λ is constant at points x with the same set of active constraints.

Let be the boolean vector $I(x) \in \Re^{ne}$ such that $I(x) = 0$ if $\lambda_i = 0$ and $I(x) = 1$ if $\lambda_i \neq 0$. It follows that $\bar{\alpha}(x)$ is linear in x for $I(x)$ constant.

For each set of design variables, we can define the collapse mechanism as the set of elements that plastify for the critical loading and causes the structure to fail. Then, $I(x)$ describes the collapse mechanism at x.

4 THE OPTIMIZATION PROBLEM

The optimization problem considered here consist on the minimization of a smooth objective function of the cross sections of the bars with smooth constraints, as well as constraints on the critical loadings computed by limit analysis, as shown in Section 3. In particular, the objective can be the structural weight and we can include constraints on stress and displacement computed by linear or nonlinear elastic analysis. Side constraints on the design variables can also be included. This problem can be expressed as a mathematical program as follows:

$$\begin{array}{ll} \text{minimize } f(x) & \\ \text{subject to:} & g(x) \leq 0 \\ & \alpha_{ad} - \bar{\alpha}(x) \leq 0 \end{array} \tag{5}$$

where $f \ : \Re^n \rightarrow \Re$ is the objective and $g \ : \Re^n \rightarrow \Re^m$ represent the inequality smooth constraints, including the side bounds.

In this model for optimal design we consider simultaneously constraints coming from elastic and limit analysis. This approach gives lighter safe structures when compared the classical elastic methodology with displacement and stress constraints. Since the feasible stresses for the limit analysis can be taken less severe than the one for the elastic analysis, the present approach gives lighter designs.

In the particular case when $f(x)$ is the structural weight, there is one loading condition and only the Limit Analysis constraint is considered, Problem (5) is equivalent to:

$$\begin{array}{ll} \text{minimize } f(x) & \\ x, T & \\ \text{subject to:} & g(x) \leq 0 \\ & B^T T - \alpha_{ad} P = 0 \\ & T - M\, L\, x \leq 0 \\ & x_{min} - x \leq 0 \end{array} \tag{6}$$

In effect, it is clear that at the optimum the collapse amplification factor $\bar{\alpha}$ will assume the allowable value. This is in some way similar to Full Stress Design. Since the weight is a linear function of x, (6) is a Linear Mathematical Program. Under appropriate assumptions, this formulation can be extended to multiple loading cases.

5 THE NUMERICAL OPTIMIZATION PROCEDURE

We present now a numerical technique to solve Problem (5) based on the Feasible Directions Interior Point Algorithm (FDIPA) described in Herskovits(1998)[15] and Herskovits(1995)[16]. FDIPA makes iterations in the primal and dual variables to solve Karush-Kuhn-Tucker first-order optimality conditions. At each iteration, a descent direction is defined by solving a linear system. In a second stage, the linear system is perturbed so as to deflect the descent direction and obtain a feasible descent direction. A line search is then performed to get a new interior point and to ensure global convergence. There are first order, Newton, and quasi Newton versions of FDIPA.

The Limit Analysis constraints are nonsmooth and FDIPA is an algorithm for smooth optimization. In this paper we present a procedure to overcome this difficulty.

Let be $I \equiv \{I_1, I_2, ..., I_r\}$ the set of all possible collapse mechanism of the structure for different values of x. We denote by $\bar{\alpha}_l(x)$ the function that represents the collapse amplification factor for the collapse mechanism I_l. Then, in Problem (5) the limit analysis constraint can be substituted by the set of constraints:

$$\alpha_{ad} - \bar{\alpha}_l(x) \leq 0, \text{ for } l = 1, 2, .., r.$$

These constraints are linear and their derivatives are given by (4). In practice, we do not include all this constraints. At each iterate x^k a limit analysis is carried out, to obtain $\bar{\alpha}(x^k)$ and the corresponding collapse mechanism $I(x^k)$. If this is the first time that this mechanism is obtained, then a new constraint $\alpha_{ad} - \bar{\alpha}_l(x) \leq 0$ is added to the already existing set of constraints of the problem. The linearized constraints are stored in the function

$$h(x) \equiv [(\alpha_{ad} - \bar{\alpha}_1(x)), (\alpha_{ad} - \bar{\alpha}_2(x)), ..., (\alpha_{ad} - \bar{\alpha}_l(x))],$$

were l is increased each time a new collapse mechanisms is obtained.

Let be λ and ϕ the Lagrange Multipliers corresponding to the general inequality and the Limit Analysis constraints respectively and $\Lambda = diag(\lambda), \Phi = diag(\phi)$. The matrix B represents a quasi - Newton approximation of the second derivative of the Lagrangian. The algorithm for Problem (5) is stated as follows :

Algorithm

Parameters. $\xi \in (0,1), \eta \in (0,1), \varphi > 0$ and $\nu \in (0,1)$.
Data. x^0 interior, $h_1(x^0) = \alpha_{ad} - \bar{\alpha}_1(x^0), 0 < \lambda \in R^m, 0 < \phi \in R, B \in R^{n\times n}$ symmetric and positive definite. Set $k = 0$ and l=1.
Step 1. Computation of a search direction.
(i) Compute (d_0, λ_0, ϕ_0) by solving the linear system

$$Bd_0 + \nabla g(x^k)\lambda_0 + \nabla h(x^k)\phi_0 = -\nabla f(x^k), \tag{7}$$

$$\Lambda \nabla g^T(x^k)d_0 + G(x^k)\lambda_0 \;=\; 0. \tag{8}$$

$$\Phi \nabla h^T(x^k)d_0 + H(x^k)\phi_0 \;=\; 0. \tag{9}$$

If $d_0 = 0$, stop.
(ii) Compute (d_1, λ_1, ϕ_1) by solving the linear system

$$Bd_1 + \nabla g(x^k)\lambda_1 + \nabla h(x^k)\phi_1 = 0, \tag{10}$$

$$\Lambda \nabla g^T(x^k)d_1 + G(x^k)\lambda_1 = -\lambda, \tag{11}$$

$$\Phi \nabla h^T(x^k)d_1 + H(x^k)\phi_1 = -\phi. \tag{12}$$

(iii) If $d_1^T \nabla f(x^k) > 0$, set

$$\rho = min[\varphi \parallel d_0 \parallel_2^2 ;\ (\xi - 1)d_0^T \nabla f(x^k)/d_1^T \nabla f(x^k)]. \tag{13}$$

Otherwise, set

$$\rho = \varphi \parallel d_0 \parallel_2^2 . \tag{14}$$

(iv) Compute the search direction

$$d = d_0 + \rho d_1, \tag{15}$$

and also

$$\bar{\lambda} = \lambda_0 + \rho\lambda_1. \tag{16}$$

$$\bar{\phi} = \phi_0 + \rho\phi_1. \tag{17}$$

Step 2. Line search.
Compute t, the first number of the sequence $\{1, \nu, \nu^2, \nu^3, ...\}$ satisfying

$$f(x^k + td) \le f(x^k) + t\eta\nabla f^T(x^k)d, \tag{18}$$

$$g(x^k + td) < 0, \tag{19}$$

and

$$h(x^k + td) \le 0. \tag{20}$$

Step 3. Updates.

(i) Set

$$x^{k+1} := x^k + td$$

If $I(x^{k+1}) \neq I_1, I_2, ..., I_l$; Set $h_{l+1} = (\alpha_{ad} - \bar{\alpha}_{l+1})$ and $l = l + 1$.

$$k = k + 1$$

and define new values for

$$\lambda > 0$$

$$\phi > 0$$

and

$$B$$

symmetric and positive definite.

(ii) Go back to Step 1. $\square$

In a similar way, an algorithm based on the Feasible Arc Interior Point Algorithm, FAIPA[17], can be obtained.

The theoretical results obtained in Herskovits(1998)[15] can be extended to the present algorithms. In particular, global convergence to Karush-Kuhn-Tucker point of the problem.

6 NUMERICAL EXAMPLES

The present method is applied to the weight minimization of three illustrative test trusses. We only consider for our tests linear elastics stress and displacements constraints together with limit analysis constraints, since this is the most interesting case for comparison with the classical structural model. In practical applications we can also include different types of smooth constraints. Local buckling conditions can be included as explained in Vanderplaats(1984)[18]. The results are obtained using 3D linear elastic pin joined bars. The required sensitivities for Linear Elastic Analysis are evaluated analytically as in Haftka et al.(1990)[20] and for Limit Analysis following the formulation described in Sections 3 and 4. Within each example three cases are considered:

Case 1) Linear elastic constraints.

Case 2) Limit and linear displacements analysis constraints.

Case 3) Limit, linear displacements and stress analysis constraints.

In all the examples the same yield stress σ_y is considered for both elastic an limit analysis. In the first case, $\sigma_{ad} = \sigma_y/\eta_{ad}$, where η_{ad} is a safety factor. If $\alpha_{ad} < \eta_{ad}$, it is easy to show that the Limit Analysis constraints are not active. We consider $\alpha_{ad} = \eta_{ad}$ in all test problems.

The designs obtained here correspond to Karush-Kuhn-Tucker point of the problem. When only limit analysis constrained are considered, it can be proved that the problem is convex. Thus, there is only one KKT point that is a global minimum. In other cases a local minimum can be obtained, however we tried with several initial design and the same solution were obtained.

Example 1: Ten bar truss, Fig 1. This structure supports one loading condition, with $p = 100000\ lb$. For the optimal design, ten variables are considered and the minimum area for each bar is $0.10\ in^2$. The material density is $0.10\ lb/in^3$, the Young modulus is $10.^7\ psi$ and the yield stress is $\sigma_y = \pm 40000\ psi$. The initial design is $x_i = 10\ in^2$ for all variables corresponding to a weight of $4196.46\ lb$. Displacement constraints are not considered in this problem. The optimal designs are shown in Table 1, when our results, for Cases 1 and 2, are compared with the ones shown in the open literature and each design variable represents the cross section of the element. In Fig.2 we show our iterations history.

Example 2: Twenty five bars 3D tower, Fig.3. This structure supports two loading conditions as shown in Table 7. The sizes of the bars are defined by eight design variables with a minimum value of $0.01\ in^2$ and a displacement limit of $2.0 in$ is imposed on all nodes. The material density is $0.10\ lb/in^3$, the Young modulus is $10.^7\ psi$ and the yield stress is $\sigma_y = \pm 60000\ psi$. The initial design is $x_i = 3\ in^2$ for all variables corresponding to a weight of $992.16\ lb$.

Example 3: Seventy two bars space truss, Fig.5. With two loading conditions described in Table 4 , the sizes of the bars are defined by sixteen design variables. The minimum value of the variables is of $0.1 in^2$. The yield stress is $\sigma_\gamma = \pm 25000 psi$. The minimum value of the displacements is $0.25 in$. The optimal designs are shown in Table 5.

7 CONCLUSIONS

The examples allow us to conclude that the mathematical model and the computational system behave in a very efficient way. They also show that the use of different models can be obtained by different weights for a same structure, however, we can not state that a model is better than another. The choice of which method should be used in the project will depend on the conditions of operation of the structure and the reliability requested in the project. The algorithm used to solve the problem of non-differentiable programming also behaves in a very efficient way in the different examples explored.

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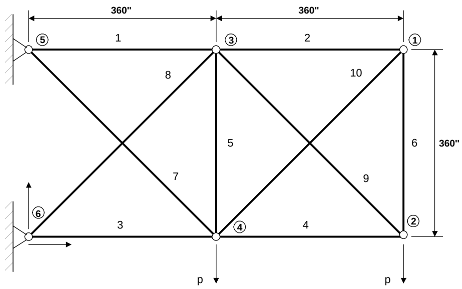

Figure 1: 10 Bars Truss

Table 1: 10 Bars Truss - Optimal Design - Cross Sections in in^2

Element	Case 1 Schmidt et al. (1976)	Present	Case 2 Kirsh (1981)	Present	Case 3 Present
1	7.938	7.9378	8.0	7.8	7.8582
2	0.1	0.1	0.1	0.1	0.1
3	8.062	8.0621	8.0	8.2	8.1418
4	3.938	3.9378	3.9	3.9	3.9
5	0.1	0.1	0.1	0.1	0.1
6	0.1	0.1	0.1	0.1	0.1
7	5.745	5.7447	5.66	5.9397	5.8574
8	5.569	5.5689	5.66	5.374	5.4563
9	5.569	5.5689	5.51	5.515	5.5154
10	0.1	0.1	0.14	0.1414	0.1414
Weight	1593.23	1593.18	1591.00	1591.20	1591.20
η_{ad}	1.60	1.60	-	-	1.60
α_{ad}	-	-	1.60	1.60	1.60

Table 2: 25 Bars 3D tower Loadings(lb)

Node	Loading 1			Loading 2		
	x	y	z	x	y	z
1	1000	10000	-5000	-	20000	-5000
2	-	10000	-5000	-	20000	-5000
3	500	-	-	-	-	-
6	500	-	-	-	-	-

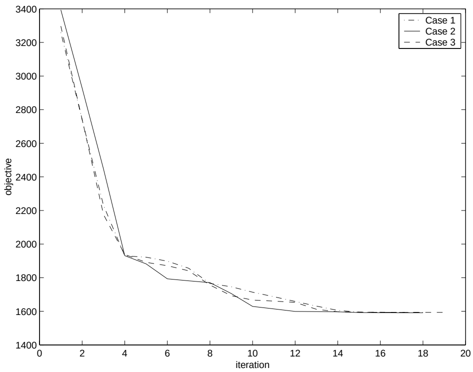

Figure 2: Iteration History for 10-Bars Truss

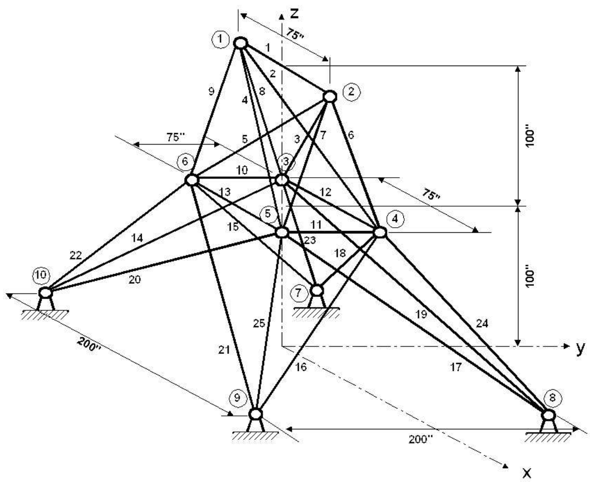

Figure 3: 25 Bars 3D tower

Table 3: 25 Bars 3D Tower - Optimal Design - Cross Sections in in^2

Variable	Element	Case 1	Case 2	Case 3
1	1	0.01	0.01	0.01
2	2-5	0.40	0.24	0.40
3	6-9	0.55	0.68	0.55
4	10-11	0.01	0.01	0.01
5	12-13	0.01	0.04	0.01
6	14-17	0.11	0.09	0.10
7	18-21	0.32	0.34	0.32
8	22-25	0.44	0.41	0.42
Weight (lb)		99.95	97.99	98.44
η_{ad}		1.7	-	1.7
$\bar{\alpha}$		-	1.7	1.7

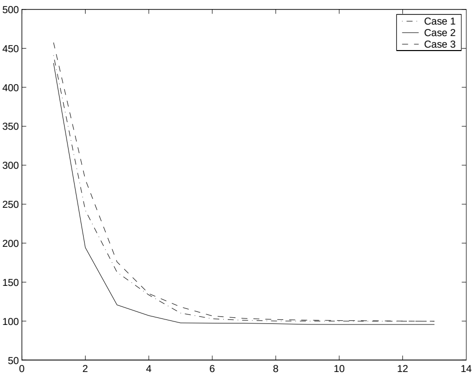

Figure 4: Iteration History for 25-Bars Truss

Table 4: 72 Bars 3D tower Loadings(lb)

Node	Loading 1			Loading 2		
	x	y	z	x	y	z
1	5000	5000	5000	0	0	-5000
2	-	-	-	0	0	-5000
3	-	-	-	0	0	-5000
4	-	-	-	0	0	-5000

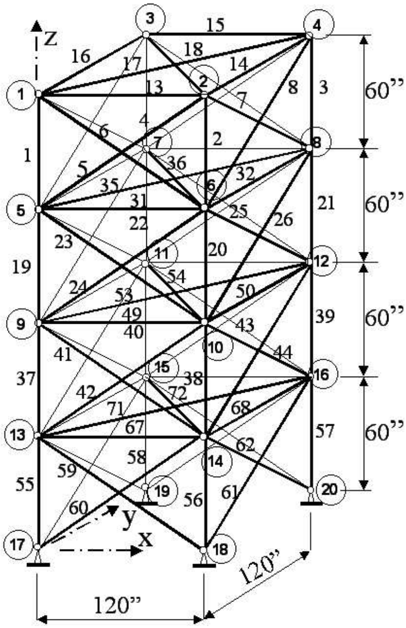

Figure 5: 72 Bars 3D tower

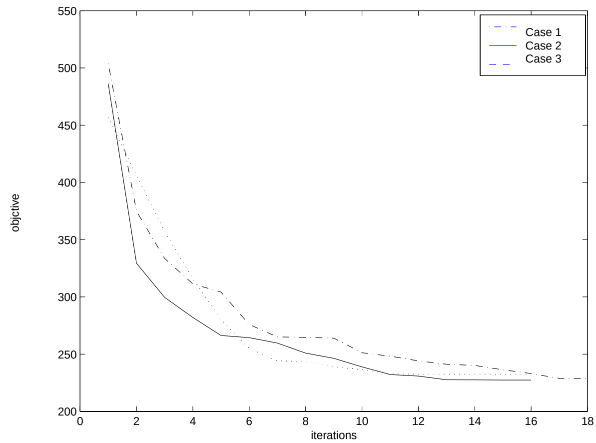

Figure 6: Iteration History for 72-Bars Truss

Table 5: 72 Bars Truss - Optimal Design - Cross Sections in in^2

Variable	Element	Case 1	Case 2	Case 3
1	1-4	0.6737	0.4438	0.6737
2	5- 12	0.3411	0.2864	0.3432
3	13-16	0.2151	0.2194	0.2129
4	17 18	0.2784	0.3173	0.2674
5	19-22	0.6885	0.7854	0.6884
6	23- 30	0.2458	0.2571	0.2460
7	31 - 34	0.1000	0.1000	0.1000
8	35 36	0.1000	0.1000	0.1000
9	37- 40	0.7128	0.6249	0.7113
10	41 - 48	0.2349	0.2550	0.2156
11	49-52	0.1000	0.1000	0.1000
12	53 54	0.100	0.1000	0.1000
13	55 - 58	1.0352	0.9417	1.0359
14	59- 66	0.2355	0.2556	0.2096
15	67 - 70	0.1000	0.1000	0.1000
16	71-72	0.1000	0.1000	0.1000
Weight (lb)		232.51	226.15	227.40
η_{ad}		3.5	-	3.5
$\bar{\alpha}$		-	3.5	3.5