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ON DESIGN SENSITIVITY IN SOME BUCKLING PROBLEMS

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Abstract. *Typical design sensitivity problems in composite materials are solved to investigate the sensitivity of the buckling load of a structure with respect to changes in design parameters, such as the angle of lamination of the composite or the volume ratio. Because design sensitivity problems can take any state as a reference state, this paper proposes to use two states to approximate sensitivity and thus cover the complete range of lamination angles between 0 and 180 degrees, together with a Hermite interpolation based on derivatives at the ends of the domain considered. The technique proves to be useful for low-order Hermite interpolation. The possibilities exceed the field of sensitivity in composites and can be applied to other problems. However, for high order Hermite interpolation it is shown that the solution beaks and leads to chaos.*

1 INTRODUCTION

Hermite interpolations have been employed in a number of problems in computational mechanics. In some cases, a low-order approximation is used (of cubic or quintic order) as shape functions in finite element formulations, and the results are good provided the response being approximated is a smooth function. High-order Hermite interpolations are used in the context of the Ritz method, in which case the number of terms required to obtain a solution may increase. In order to obtain the coefficients of the Hermite expansion, one needs to establish a set of constraints that may arise from the differential equations in the domain or from boundary conditions.

This paper addresses problems arising from design sensitivity of buckling and post-buckling problems, for which perturbation expansions have been the preferred technique of analysis [1-3]. However, Hermite interpolations can also be used, in which all terms in a Hermite interpolation may be specified by means of conditions involving derivatives of the function at the end of the domain investigated. Two points are chosen to expand a solution using known values of the function and its derivatives, and a Hermite interpolation rather than a classical perturbation approach has been chosen because the results seem to be advantageous.

In order to generalize the approach, a problem with higher-order derivatives available at the ends of an interval has been studied. Some perplexing behavior of the solution was found.

2 DESIGN SENSITIVITY IN THE BUCKLING OF THIN-WALLED COMPOSITE COLUMNS

2.1 Perturbation expansion with respect to a reference state

In a regular perturbation expansion, one knows the values of a function $f(x)$ and its derivatives at a point, and attempts to find the values of $f(x+\Delta x)$ at neighboring points. Such perturbation approximations find a number of applications in mechanics, including the theory of elastic stability [6], sensitivity analysis in buckling problems [3], nonlinear eigenvalue problems, dynamics [4] and several others. The perturbation expansions can be a reasonably good approximation for small values of Δx, but they become less accurate as Δx increases.

Consider, for example, a nonlinear sensitivity analysis of the buckling problem of a laminated composite structure. One may compute design sensitivity of buckling loads and post-buckling curvature with respect to changes in the lamination angle of the laminate. A full account of the analytical sensitivity formulation may be found elsewhere in the literature [1, 2]. Because of the nature of the problem, any design configuration may be taken as a reference state and the sensitivity can be expanded from this state to others in the neighborhood.

Figure 1 shows results for a 3”×6” channel section column under axial load, for which detailed data is given elsewhere [1]. Consider a reference state A, for which the lamination angle is $\theta = 0°$ (the case with $\theta = 90°$ is entirely similar), and the perturbation approximation becomes:

$$f(\xi) = f^A + \xi f^A + 1/2\xi^2 df^A / d\xi + \ldots \tag{1}$$

where f is the critical load, ξ is the perturbation parameters (the lamination angle θ measured with respect to $\theta = 0°$), and the derivatives with respect to the perturbation parameter ξ are evaluated at $\xi = 0$. The expansion is seen to diverge from the exact solution for $\theta > 25°$, with a significant error at $\theta = 45°$.

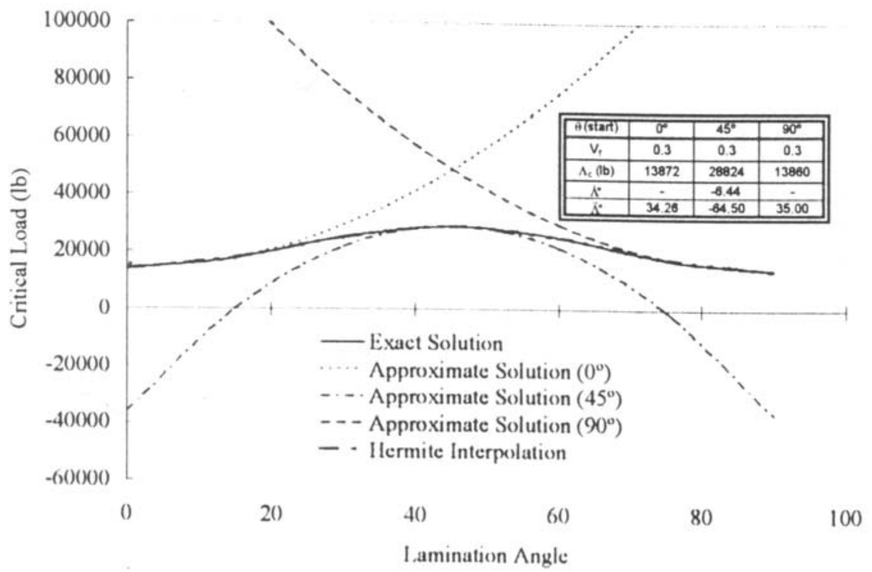

Figure 1: Sensitivity of the critical state of a 3×6 inches channel section column, for changes in lamination angle.

Instead of using A as a reference state, one may choose state B (for $\theta = 45°$), for which a new perturbation approximation similar to equation (1) is constructed. The result is shown in Figure 1, and again the approximation is good for changes in lamination angle of the order of *15-20°*, but the errors for $\theta = 0°$ are very high.

2.2 Two-point perturbation expansion in sensitivity analysis

To improve the accuracy of $f(x)$ in the interval AB, one can take advantage of the fact that values of the derivatives of f have already been computed at the ends of the interval. Thus, rather than constructing individual approximations at A and B, one can employ a Hermite interpolation using derivatives. Then, a cubic interpolation equation $f(\xi)$ can be expressed using four parameters as follows:

$$f(\xi) = a_0 + a_1\,\xi + a_2\,\xi^2 + a_3\,\xi^3 \tag{2}$$

subject to conditions at the ends of the interval. Inverting the expression (2) leads to

$$f(\xi) = \Phi_1 f^A + \Phi_2 f^B + \Phi_3\, df^A/d\,\xi + \Phi_4\, df^B/d\,\xi + \tag{4}$$

The functions ϕ are Hermite polynomials and can be written as follows:

$$\Phi_1=1-3\,\xi^2+2\,\xi^3 \qquad \Phi_2=3\,\xi^2-2\,\xi^3 \tag{3}$$

$$\Phi_3=-AB(1-\xi)^2 \qquad \Phi_4=-AB(\xi^2-\xi)$$

Clearly, this is a cubic interpolation of values inside an interval *AB*.

The results for this interpolation are shown in Figure 1, and it is seen that they coincide very well with the exact solution of this problem. The information required to compute the Hermite interpolation is the same employed in the two-point perturbation expansion, i.e. up to second order derivatives of *f* at the ends of the interval.

Similar behavior is found for the sensitivity of the curvature of the post-buckling equilibrium path, as shown in Figure 2.

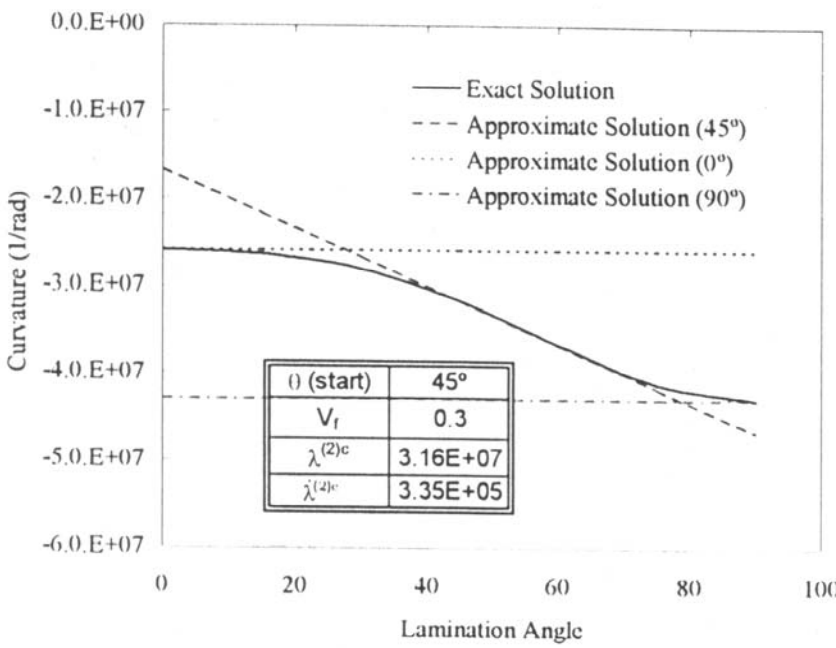

Figure 2: Post buckling sensitivity analysis of a 3×6 inches channel section column, for changes in lamination angle.

3 HIGHER-ORDER INTERPOLATIONS

Similar to what was done in the previous sections, we now consider that the function *f* and its derivatives are available at the ends of the interval $x = [0, 1]$. But rather than having a limited number of derivatives available at the ends of the interval, we now consider the case with any number of known derivatives at the ends. The resulting expression is

$$f(\xi) = a_n\,\xi^n \qquad for \qquad n = 0..N \tag{5}$$

subject to constraints.

3.1 Statement of the problem

Consider an equilibrium path similar to that found in neutral post-buckling behavior. One such possibility is illustrated in Figure 3.

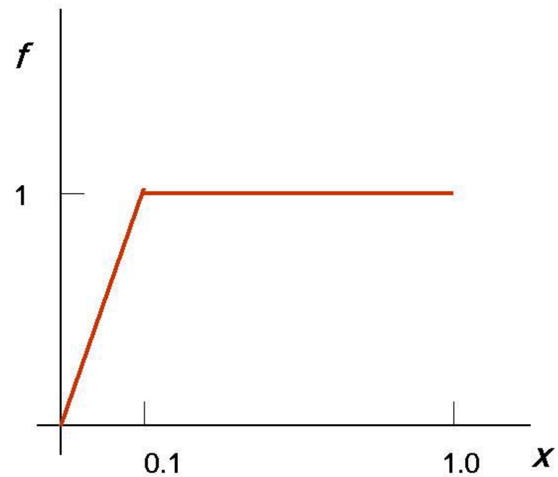

Figure 3: Equilibrium path for neutral post-buckling behavior.

The initial slope is assumed to be $\Delta f/\Delta x = 1/0.1 = 10$. For $x = 1$, we get $f = 1$. Furthermore, the other derivatives are

$$d^n f/dx^n = 0 \quad at \quad x = 0 \quad and \quad x = 1 \tag{8}$$

The path bends at the critical state with a very high curvature, in order to accommodate to the imposed end-conditions. What order of polynomial (or perturbation approximation) would be required to follow such a path?

To represent the path of Figure 3, let us assume a polynomial expansion

$$f(x) = a_i\, x_i \quad for \quad i = 0, .. N \tag{9}$$

under the conditions

$$f(0) = 0 \qquad f(1) = 1 \qquad df/dx(0) = 10 \tag{10}$$

And all the other derivatives are zero, as indicated in equation (8).

3.2 Poor approximation with few terms

Plots of the resulting curves are shown in Figure 4 for approximations including from 1 to 11 terms. The results show that for $N = 2$, the slope at $x = 1$ is not horizontal. For $N = 3$ it becomes horizontal (cubic approximation) but the load f exceeds 1. As we increase N (values of $N = 5, 7, 9$ and 11), the curves become flatter at $x = 1$, but increase the peak value away from 1. Such results have been computed using 12 digits accuracy and the symbolic manipulator MAPLE.

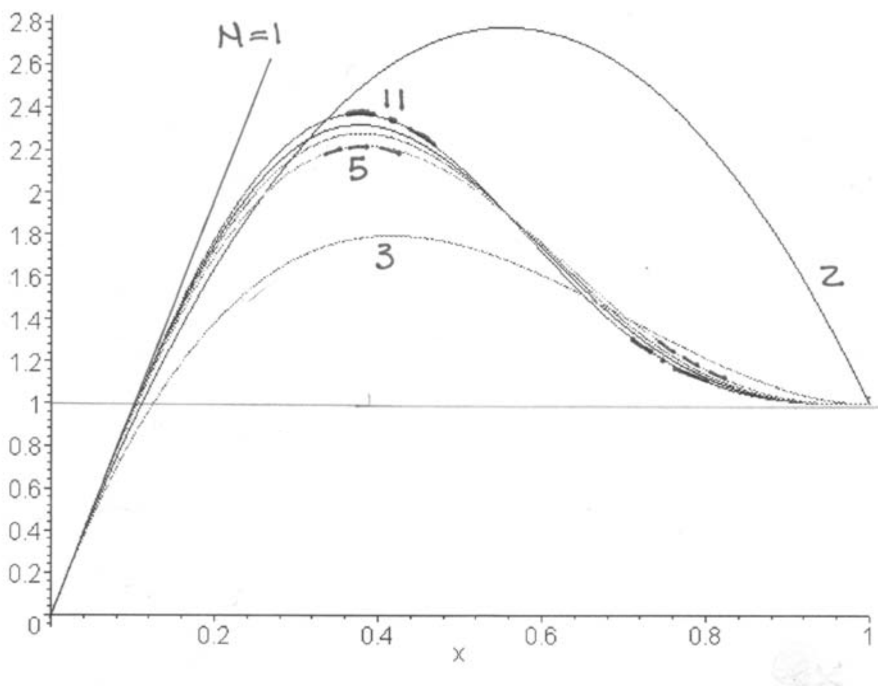

Figure 4: Equilibrium path (load versus displacement) of Figure 3 using 1, 2, 3, 5, and 11 terms in the Hermite expansion.

3.3 Perplexing behavior for higher order terms

A much higher polynomial approximation would be required to approximate the desired curve, perhaps *N* of the order of 100. We wrote a small program using MAPLE V v.5.1 to solve and plot the results for any number of terms. The order of derivatives that are constrained at the ends of the interval is half the order of the polynomial.

No noise is detected for $N = 30$ terms (Figure 5). This is the limit for which all constraints are satisfied. For $N = 36$, a small noise starts at $x = 1$. As we increase the number of terms to 40 (Figure 6), the curve does not satisfy the constraints at $x = 1$, but it still satisfies those at $x = 0$. And a completely wild solution is obtained for $N = 50$ and $N = 100$ in Figure 7.

The occurrence of this numerical behavior depends on the accuracy of the computations, i.e. on the number of digits employed to carry out the computations. As the number of constraints increases to 30, then the number of digits required to avoid chaos is 12. But for 48 terms, 20 digits are needed; and for 100 constraints the solution requires 70 digits! This is an indication of chaotic behavior, since even extremely small changes in the accuracy of the computations lead to large changes in the resulting curve being represented.

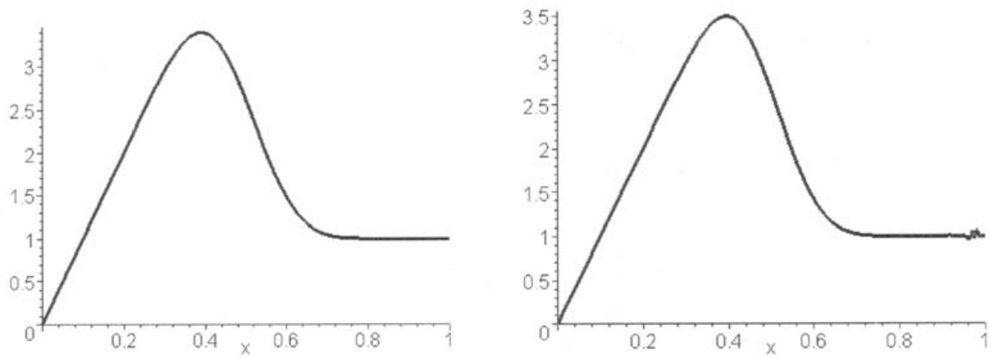

Figure 5: Equilibrium path for the problem of Figure 3 using 30 and 36 terms in the Hermite expansion

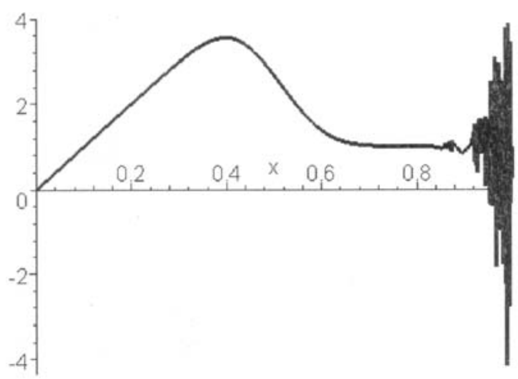

Figure 6: Equilibrium path for the problem of Figure 3 using 40 terms in the Hermite expansion

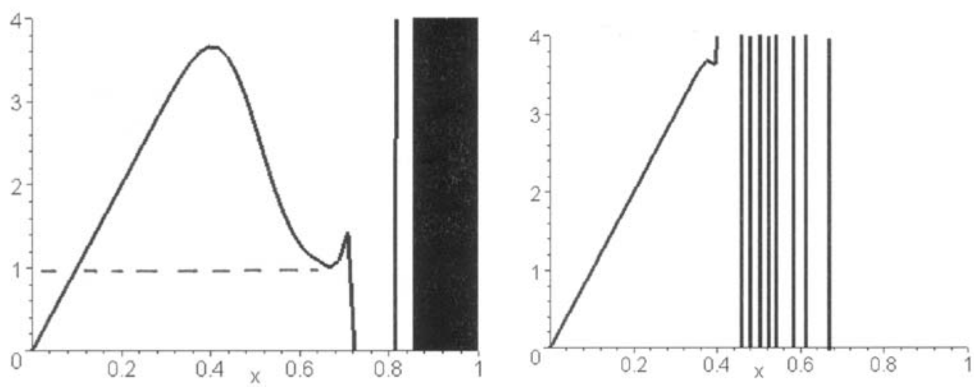

Figure 7: Equilibrium path for the problem of Figure 3 using 50 and 100 terms in the Hermite expansion

4 DISCUSSION AND CONCLUSIONS

Hermite interpolations have been studied in this short paper, with an interest arising from the expansion of design sensitivity solutions in buckling and post-buckling problems.

Two problems were investigated in this paper: First, a design sensitivity analysis, for which the sensitivity curve was first obtained using perturbation expansions; then computed using two separate perturbation expansions; and finally obtained using Hermite interpolation by means of derivatives of the function at the two ends of the interval. An important feature in this problem is that there is no singularity in the interval investigated, and just a few terms are needed to obtain a good estimate of sensitivity.

The second problem was a bilinear function in the interval of interest, and was directly tackled by Hermite interpolation using values at the ends of the interval, since both lines have zero derivatives up to any order and the Hermite interpolation can be taken up to any degree desired.

Two interesting aspects were detected in the second problem. The approximation does not resemble the target function: It seems that this effect is due to the presence of a singularity at the intersection between the two lines, so that the best that can be obtained is a smooth curve but with a different shape.

The approximation collapsed for interpolations including more than 30 terms. High amplitude oscillations are produced at the far end of the interval, with oscillation amplitudes much higher than the actual values of the function being investigated. Furthermore, small changes in the number of terms used had large consequences in the response. The results display a chaotic behavior, in much the same way as reported by Thompson and Stewart [5] for chaos arising in iterative techniques. The practical implications of this perplexing behavior of the approximation are still not clear to the author.

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