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A MODEL FOR SHEAR DEFORMABLE CURVED BEAM MADE OF FUNCTIONALLY GRADED MATERIALS

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Abstract. In this paper the Heillinger-Reissner principle is employed to derive a new model for a curved beam made of functionally graded materials. The model is developed on the assumptions that the shear deformability is not negligible. The Hellinger-Reissner principle can be handled in order to derive the motion equations together with a consistent description of the constitutive equations. This leads to obtain the shear coefficients of the beam theory as an inherent part of the model deduction, thus avoiding the imposition of shear coefficients arbitrarily taken from other approaches, as one can see in many papers of the open literature.

Despite the technological importance of new materials, it is interesting to remark that curved beam models for the aforementioned graded materials are not available under the conception of a one-dimensional beam theory. On the other hand, this model contains the curved model for isotropic materials and particular cases for laminated composite beams.

A finite element procedure is employed in order to solve the motions equations for free vibrations problems with or without the presence of initial stresses. Different types of laws of graded properties are tested. Parametric studies and comparisons with analytical solutions are performed as well.

1 INTRODUCTION

During the past twenty years, strategic and high technology industries, such as defense, aerospace or automotive industries demanded new advanced materials in order to maintain or increase the leadership in the production of high competitive goods. Designers have been always claiming for materials that combine in a unified fashion, the best properties of metals and ceramics, that is, the stiffness, electrical conductivity and machinability of metals and the high strength, low density and high temperature resistance of ceramics. During the past ten or twelve years these kind of advanced materials are becoming no longer experimental specimens in laboratories but a well developed reality. Functionally graded materials (FGM) are just an example of such advanced materials. Its name was settled in the beginning of the nineties, in Japan, associated with a particular manufacturing process. The variation in percentage of the material constituents (basically from a ceramic to a metal) can be arranged in such a way to create a new material with graded properties in spatial directions. The underlying ideas about a material with graded properties reflects the notion that such material can avoid the presence of certain undesirable behaviors such as thermal stress gradients or residual stresses among others, which are common in the interface of ceramic and metallic materials.

Many papers have been devoted to study shells and solids constructed with FGM such as the works carried out by Reddy and Chin (1998), Reddy (2000), Praveen and Reddy (1998), El-Abbasi and Meguid (2000) and Kitipornchai et al. (2004) among others. The recent works of Chakraborty et al. (2003), Goupee and Vel (2006), Ding et al. (2007) and Lu et al. (2008), among others can be considered the most relevant for functionally graded straight beam. In these papers different laws defining the graded properties of the beams have been employed. The gradation laws can be of the exponential type or a power law or any other with "ad-hoc" purposes. Many of the aforementioned papers introduce a three dimensional or a two dimensional complex model. On the other hand there are quite a few papers devoted to study functionally graded curved beams. Dryden (2007) carried out a study on a curved beam by means of an approximation to a two dimensional description based in the hypothesis of plane stresses. Shafiee et al. (2006) developed a model for buckling analysis of curved beams made of FGM; however this model do not considered shear flexibility and dynamic problems were not studied.

It has to be noted that, despite its technological interest, no studies on the dynamics of curved beams made of FGM have been performed in the past years according to the knowledge of the authors. Thus, the aim of the present work is intended to be a contribution on the subject. The curved beam model is developed, taking into account the shear flexibility of the structure in the in-plane motion, through the employment of the Heillinger-Reissner variational principle together with a displacement field accounting for linear and second order terms. Also a state of arbitrary initial stresses is considered. The use of the Heillinger-Reissner variational principle leads to the motion equations and constitutive equations in a unified fashion. The constitutive equations are derived consistently avoiding the use of arbitrary taken shear factors as it appears in recent papers (Chakraborty et al., 2003 and Oh et al., 2003). A four-node isoparametric element is employed to discretize the motion equations and vibrations patterns of curved beams with different gradation properties are calculated. An alternative exact solution (based in a power series methodology) of the motion equations for the free vibration problem is also employed for comparison and validation purposes.

2 MODEL DEVELOPMENT

In Figure 1 one can see the structural model for the curved beam. In order to develop the model, the following hypotheses are performed:

a) The cross-section is rigid in its own plane, i.e. in the *ZX* plane.
b) Only in-plane motions are considered (i.e. in the *YX* plane).
c) The shear flexibility is taken into account.
d) Power, exponential, or any other law for the graded materials can be employed across the inner and outer radii.
e) No approximation is performed in the curvature terms.
f) An arbitrary in-plane state of initial stresses is also considered.

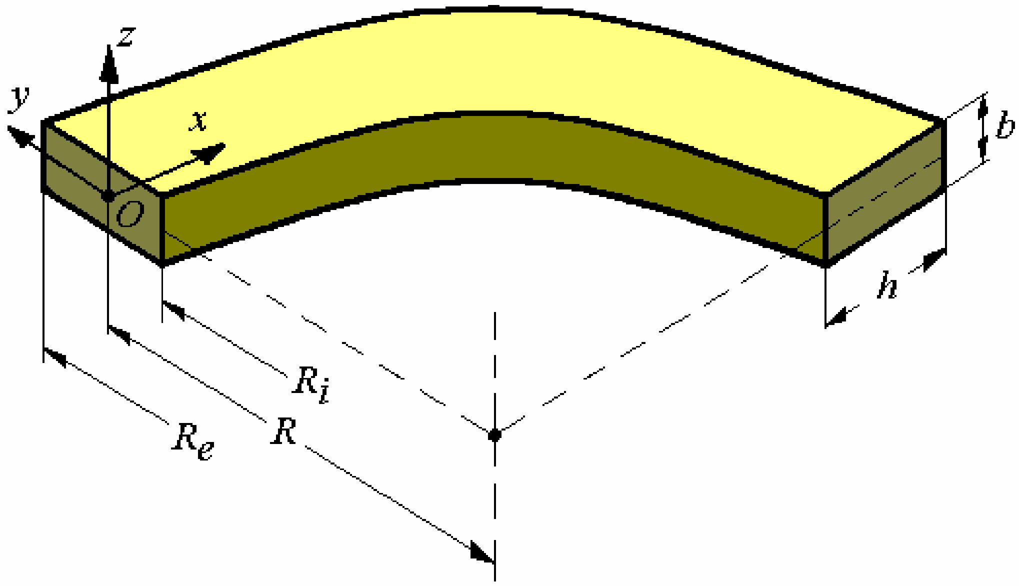

Figure 1: Structural model.

The reference system $\{O{:}xyz\}$ is located in the geometric center of the cross-section. The beam can be configured with graded properties in such a way that the core is mainly metallic and the surfaces are mainly ceramic or with a graded variation from a metallic surface at the inner radius R_i and a ceramic surface at the outer radius R_e. In the first case the properties can vary according to the Eq. (1a), but in the second case the properties can vary according to Eq. (1b) or (1c) for exponential or power law, respectively.

$$\mathcal{P}(y)=\mathcal{P}_{M^+}\left[\Lambda_M+(1-\Lambda_M)\left|\frac{2y}{h}\right|^n\right],\quad \text{with } \Lambda_M=\frac{\mathcal{P}_{M^-}}{\mathcal{P}_{M^+}} \tag{1a}$$

$$\mathcal{P}(y)=\mathcal{P}_{M^+}e^{\left[-\Lambda_\delta\left(1-\frac{2y}{h}\right)\right]},\quad \text{with } \Lambda_\delta=\frac{1}{2}Log\left[\frac{\mathcal{P}_{M^+}}{\mathcal{P}_{M^-}}\right] \tag{1b}$$

$$\mathcal{P}(y)=\mathcal{P}_{M^+}\left[\Lambda_M+(1-\Lambda_M)\left(\frac{y}{h}+\frac{1}{2}\right)^n\right],\quad \text{with } \Lambda_M=\frac{\mathcal{P}_{M^-}}{\mathcal{P}_{M^+}} \tag{1c}$$

Where, $\mathcal{P}(y)$ denotes a typical material property (i.e., density ρ or Young's modulus E or transversal elastic modulus G, among others). $\mathcal{P}_{M^+}$ and $\mathcal{P}_{M^-}$ define the properties of the outer (normally ceramic) and inner materials (normally metallic).

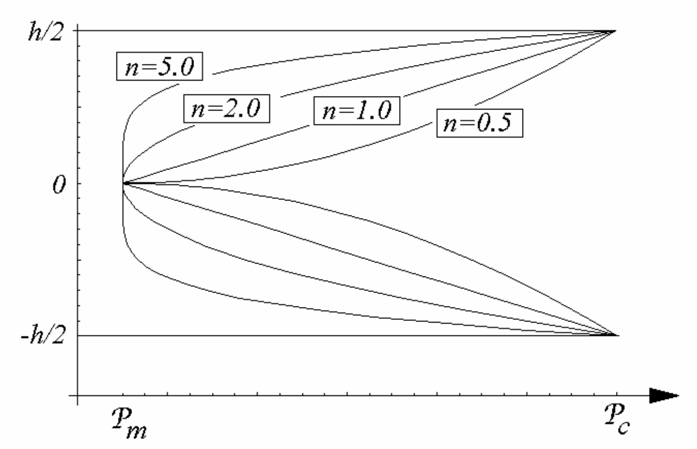

Figure 2: Variation of a typical property with the power exponent *n*.

In Figure 2 one can see the variation with respect to the power exponent *n* of a typical property along the radial coordinate according to, for example, Eq. (1a). In this case the ceramic component is located at the inner and outer surfaces, whereas the metallic component is placed in the core. Notice the proportional increase of the metallic component with the increase of exponent *n*. In the limiting case, when $n \rightarrow \infty$, the whole beam is made by a metallic constituent.

2.1 The Heillinger-Reissner Functional

The variational principle of Heillinger-Reissner for an in-plane curved beam subjected to initial stresses may be presented in the following form (see Washizu, 1974):

$$\begin{aligned}&\int_V \left[\sigma_{xx}\delta\varepsilon_{xx}^L + 2\sigma_{xy}\delta\varepsilon_{xy}^L\right]dV + \int_V \left[\sigma_{xx}^{(0)}\delta\varepsilon_{xx}^{NL} + 2\sigma_{xy}^{(0)}\delta\varepsilon_{xy}^{NL}\right]dV - \\ &- \int_{S_a}\left[\overline{T}_x^{(0)}\delta u_x^{NL} + \overline{T}_y^{(0)}\delta u_y^{NL}\right]dS - \int_V\left[\overline{X}_x^{(0)}\delta u_x^{NL} + \overline{X}_y^{(0)}\delta u_y^{NL}\right]dV - \\ &- \int_{S_a}\left[\overline{T}_x\delta u_x^L + \overline{T}_y\delta u_y^L\right]dS - \int_V\left[\overline{X}_x\delta u_x^L + \overline{X}_y\delta u_y^L\right]dV + \\ &+ \int_V\left[\rho\ddot{u}_x^L\delta u_x^L + \rho\ddot{u}_y^L\delta u_y^L\right]dV = 0\end{aligned} \tag{2a}$$

$$\int_V\left[\left(\varepsilon_{xx}^L - \frac{\sigma_{xx}}{E}\right)\delta\sigma_{xx} + \left(2\varepsilon_{xy}^L - \frac{\sigma_{xy}}{G}\right)\delta\sigma_{xy}\right]dV = 0 \tag{2b}$$

As it can be seen, Eq. (2a) corresponds to the common principle of virtual work; however (2b) corresponds to the variational form of the complementary energy. The above variational principle is subjected to the next constraint equation corresponding to the initial configuration due to the initial stresses:

$$\int_V\left[\sigma_{xx}^{(0)}\delta\varepsilon_{xx}^L + 2\sigma_{xy}^{(0)}\delta\varepsilon_{xy}^L\right]dV - \int_V\left[\overline{X}_x^{(0)}\delta u_x^L + \overline{X}_y^{(0)}\delta u_y^L\right]dV - \int_{S_a}\left[\overline{T}_x^{(0)}\delta u_x^L + \overline{T}_y^{(0)}\delta u_y^L\right]dS = 0 \tag{3}$$

In Eq. (2a), (2b) and (3), σ_{ij} and $\sigma_{ij}^{(0)}$ are the incremental and initial stresses, $\overline{X}_i$ and $\overline{X}_i^{(0)}$ are incremental and initial volume forces, $\overline{T}_i$ and $\overline{T}_i^{(0)}$ are incremental and initial surface forces acting in the domain S_a; ρ, E and G are material properties corresponding to the density, longitudinal elasticity modulus and shear elasticity modulus, respectively. ε_{ij}^{L} and ε_{ij}^{NL} are the linear and non-linear strain components given by the following form (Piovan and Cortinez, 2007):

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left(\frac{\partial u_j^L}{\partial x_i} + \frac{\partial u_i^L}{\partial x_j}\right),\ \varepsilon_{ij}^{L} \cong \frac{1}{2}\left(\frac{\partial u_h^L}{\partial x_i}\frac{\partial u_h^L}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial u_j^{NL}}{\partial x_i} + \frac{\partial u_i^{NL}}{\partial x_j}\right) \tag{4}$$

where u_i^L and u_i^{NL} are linear and non-linear (or second order) displacement components, whereas i, j and h are typical indexes, according to the Einstein's notation.

2.2 The displacement field and for a curved beam

Taking into account aforementioned assumptions (a)–(c) it is possible to develop the displacement field of the in-plane motion, for an arbitrary point of a curved beam including first- and second-order terms of rotational parameters, in the following form:

$$u_x(x,y,t) = u_x^L + u_x^{NL} = u_{xc} - y\left(\theta_z - \frac{u_{xc}}{R}\right) \tag{5a}$$

$$u_y(x,y,t) = u_y^L + u_y^{NL} = u_{yc} - \frac{y}{2}\left(\theta_z(x,t) - \frac{u_{xc}(x,t)}{R}\right)^2 \tag{5b}$$

where u_{xc} and u_{yc} are the displacements of the reference point O, θ_z is the rotation parameter for bending.

Employing (5a) and (5b) in (4), and after a few algebraic steps, the strain components can be defined in terms of the displacement variables u_{xc}, u_{yc} and θ_z as:

$$\varepsilon_{xx}^{L} = \left(\varepsilon_N - y\,\varepsilon_M\right)\mathcal{F} \tag{6a}$$

$$2\varepsilon_{xy}^{L} = \varepsilon_Q\,\mathcal{F} \tag{6b}$$

$$\varepsilon_{xx}^{NL} = \left[\left(\varepsilon_N - y\,\varepsilon_M\right)^2 + \left(\varepsilon_Q + \frac{\varepsilon_B}{\mathcal{F}}\right)^2\right]\frac{\mathcal{F}^2}{2} - \frac{y}{R}\left(\varepsilon_B\right)^2\frac{\mathcal{F}}{2} \tag{6c}$$

$$2\varepsilon_{xy}^{NL} = -\varepsilon_B\left(\varepsilon_N + y\,\varepsilon_M\right)\frac{\mathcal{F}}{2} \tag{6d}$$

where:

$$\varepsilon_N = \left(\frac{\partial u_{xc}}{\partial x} + \frac{u_{yc}}{R}\right),\ \varepsilon_M = \left(\frac{\partial \theta_z}{\partial x} - \frac{1}{R}\frac{\partial u_{xc}}{\partial x}\right),\ \varepsilon_Q = \left(\frac{\partial u_{yc}}{\partial x} - \theta_z\right) \tag{7a-c}$$

$$\varepsilon_B = \left(\theta_z - \frac{u_{xc}}{R}\right) \tag{7d}$$

$$\mathcal{F} = \frac{R}{R+y} \tag{7e}$$

In Eq. (7a-c), ε_N, ε_M and ε_Q may be regarded as generalized deformations. That is ε_N identify the generalized axial deformation, ε_M is the bending deformation and ε_Q is the bending shear deformation.

2.3 Motion Equations

Substituting Eq (5a)-(5b) and (6a)-(6d) into Eq (2a) and integrating in the corresponding domains one gets the following Equation:

$$\mathcal{L}_K + \mathcal{L}_G + \mathcal{L}_M + \mathcal{L}_P = 0 \tag{8}$$

where $\mathcal{L}_K$ is the virtual work due to internal forces, $\mathcal{L}_G$ is the virtual work due to initial stresses and of applied volume and surface initial forces due to second-order displacements, $\mathcal{L}_M$ is the virtual work of inertial forces and $\mathcal{L}_P$ is virtual work of applied volume forces and surface forces. This last term may include, as a limiting case, applied point forces or beam forces.

The terms of the virtual work Eq (8) can be written in the following compact form:

$$\mathcal{L}_K = \int_0^L \left\{\delta\Delta^{(E)}\right\}^T \left\{Q^{(E)}\right\} dx \tag{9a}$$

$$\mathcal{L}_G = \int_0^L \left\{\delta\Delta^{(G)}\right\}^T \left[T_E^{(0)}\right]\left\{\Delta^{(G)}\right\} dx - \int_0^L \{\delta U\}^T \left[C_X^{(0)}\right]\{U\} dx - \{\delta U\}^T \left(\left[C_T^{(0)}\right] + \left[C_D^{(0)}\right]\right)\{U\} \tag{9b}$$

$$\mathcal{L}_M = \int_0^L \{\delta U\}^T [\mathcal{M}]\{\ddot{U}\} dx \tag{9c}$$

$$\mathcal{L}_P = -\int_0^L \{\delta U\}^T \{\mathcal{P}\} dx - \{\delta U\}^T \left\{\overline{\mathcal{T}}\right\}\Big|_{x=0,L} \tag{9d}$$

where $\{U\}$ is the vector of beam displacements, $\{\ddot{U}\}$ is the vector of accelerations and $\{\delta U\}$ is the corresponding vector of variational displacements, whereas $\left\{\Delta^{(E)}\right\}$ and $\left\{\Delta^{(G)}\right\}$ are vectors of generalized deformations given by:

$$\{U\} = \left\{u_{xc}, u_{yc}, \theta_z\right\}^T \tag{10a}$$

$$\left\{\Delta^{(E)}\right\} = \left\{\varepsilon_N, -\varepsilon_M, \varepsilon_Q\right\}^T \tag{10b}$$

$$\left\{\Delta^{(G)}\right\} = \left\{\varepsilon_N, -\varepsilon_M, \varepsilon_Q, \varepsilon_B\right\}^T \tag{10c}$$

The vector $\left\{Q^{(E)}\right\}$ introduced in Eq. (9a) contains the internal forces or beam stress resultants:

$$\left\{\Delta^{(E)}\right\} = \left\{N_X, M_Z, Q_Y\right\}^T \tag{11}$$

where N_X is the axial (or circumferential) force, M_Z is the bending moment and Q_Y is the shear force. These beam stress resultants are defined as follows:

$$\left\{N_X, M_Z, Q_Y\right\} = \int_A \left\{\sigma_{xx}, y\sigma_{xx}, \sigma_{xy}\right\} dA \tag{12}$$

The matrix $\left[T_E^{(0)}\right]$ of initial beam stress resultants, introduced in (9b), is written in the following form:

$$\left[T_E^{(0)}\right] = \int_A \begin{bmatrix} \sigma_{xx}^0 \mathcal{F} & y\sigma_{xx}^0 \mathcal{F} & 0 & -\sigma_{xy}^0/2 \\ y\sigma_{xx}^0 \mathcal{F} & y^2\sigma_{xx}^0 \mathcal{F} & 0 & y\sigma_{xy}^0 \\ 0 & 0 & \sigma_{xx}^0 \mathcal{F} & \sigma_{xx}^0 \\ -\sigma_{xy}^0/2 & y\sigma_{xy}^0 & \sigma_{xx}^0 & \sigma_{xx}^0 \end{bmatrix} dA \tag{13}$$

The matrices $\left[C_X^{(0)}\right]$, $\left[C_T^{(0)}\right]$ and $\left[C_D^{(0)}\right]$ introduced in (9b) are defined as follows:

$$\left[C_X^{(0)}\right] = \int_A \begin{bmatrix} -\dfrac{y\overline{X}_y^{(0)}}{R^2} & 0 & \dfrac{y\overline{X}_y^{(0)}}{R} \\ 0 & 0 & 0 \\ \dfrac{y\overline{X}_y^{(0)}}{R} & 0 & -y\overline{X}_y^{(0)} \end{bmatrix} dA \tag{14a}$$

$$\left[C_T^{(0)}\right] = \int_{S_a} \begin{bmatrix} -\dfrac{y\overline{T}_y^{(0)}}{R^2} & 0 & \dfrac{y\overline{T}_y^{(0)}}{R} \\ 0 & 0 & 0 \\ \dfrac{y\overline{T}_y^{(0)}}{R} & 0 & -y\overline{T}_y^{(0)} \end{bmatrix} dA \tag{14b}$$

$$\left[C_D^{(0)}\right] = \begin{bmatrix} -\dfrac{y_D\overline{F}_y^{(0)}}{R^2} & 0 & \dfrac{y_D\overline{F}_y^{(0)}}{R} \\ 0 & 0 & 0 \\ \dfrac{y_D\overline{F}_y^{(0)}}{R} & 0 & -y_D\overline{F}_y^{(0)} \end{bmatrix} \tag{14c}$$

where $\overline{X}_y^{(0)}$ and $\overline{T}_y^{(0)}$ are the volume and surface forces in the radial direction, respectively; and $\overline{F}^{(0)} = \left\{\overline{F}_x^{(0)}, \overline{F}_y^{(0)}\right\}$ is a generic point-load applied at the point $D = \{x_D, y_D\}$. Eq. (14c) can be also obtained from volume forces employing Dirac-Delta operators (Piovan and Cortínez, 2007).

In Eq. (9c) the matrix $[\mathcal{M}]$ is given by:

$$[\mathcal{M}] = \begin{bmatrix} J_{11}^{\rho} + \dfrac{2J_{12}^{\rho}}{R} + \dfrac{J_{22}^{\rho}}{R^2} & 0 & -J_{12}^{\rho} - \dfrac{J_{22}^{\rho}}{R} \\ 0 & J_{11}^{\rho} & 0 \\ -J_{12}^{\rho} - \dfrac{J_{22}^{\rho}}{R} & 0 & J_{22}^{\rho} \end{bmatrix} \tag{15}$$

where:

$$J_{ij}^{\rho} = \int_A \rho(y)\left(\overline{g}_i \cdot \overline{g}_j\right)\frac{dA}{\mathcal{F}}, \quad \text{with} \quad \overline{g} = \{1, y\} \tag{16}$$

In Eq. (9d) the vectors of forces $\{\mathcal{P}\}$and $\{\mathcal{T}\}$ are defined in the follwing form:

$$\{\mathcal{P}\} = \int_A \left\{\frac{\overline{X}_x}{\mathcal{F}}, \overline{X}_y, -y\overline{X}_x\right\}\frac{dA}{\mathcal{F}} \tag{17a}$$

$$\left\{\overline{\mathcal{T}}\right\} = \left\{\left(\overline{N}_X + \frac{\overline{M}_Z}{R}\right), \overline{Q}_Y, -\overline{M}_Z\right\} \tag{17d}$$

where $\overline{N}_X$, $\overline{M}_Z$ and $\overline{Q}_Y$ are beam stress resultants applied at the ends.

The differential equations of motion and related boundary conditions can be obtained by substituting Eq. (5), (6), (10), (11), (13), (14), (15) and (17) into Eq. (9) and then performing in Eq. (8) the conventional procedures of the variational calculus. Then, the differential equations of motion are:

$$-N'_X+\frac{M'_Z}{R}+\mathcal{D}^{(0)}_{10}+\mathcal{M}_1(x)-\mathcal{P}_1(x,t)=0 \tag{18a}$$

$$-Q'_Y+\frac{N_X}{R}+\mathcal{D}^{(0)}_{20}+\mathcal{M}_2(x)-\mathcal{P}_2(x,t)=0 \tag{18b}$$

$$M'_Z-Q_Y+\mathcal{D}^{(0)}_{30}+\mathcal{M}_3(x)-\mathcal{P}_3(x,t)=0 \tag{18c}$$

subjected to the boundary conditions:

$$-\left(\overline{N}_X+\frac{\overline{M}_Z}{R}\right)+\left(N_X+\frac{M_Z}{R}\right)+\mathcal{D}^{(0)}_{11}=0 \quad \text{or} \quad \delta u_{xc}=0 \tag{19a}$$

$$-\overline{Q}_Y+Q_Y+\mathcal{D}^{(0)}_{21}=0 \quad \text{or} \quad \delta u_{yc}=0 \tag{19b}$$

$$\overline{M}_Z-M_Z+\mathcal{D}^{(0)}_{31}=0 \quad \text{or} \quad \delta\theta_z=0 \tag{19c}$$

In Eq. (18) the following definitions are performed:

$$\mathcal{D}^{(0)}_{10}=\frac{Q^{(0)}_Y}{2R}\varepsilon_N-\frac{N^{(0)}_X}{R}\varepsilon_B+\frac{I^{(0)}_4}{R}\varepsilon_M-\frac{N^{(0)}_X}{R}\varepsilon_Q-\frac{\partial}{\partial x}\left[\mathcal{D}^{(0)}_{11}\right] \tag{20a}$$

$$\mathcal{D}^{(0)}_{20}=\frac{I^{(0)}_1}{R}\varepsilon_N-\frac{Q^{(0)}_Y}{2R}\varepsilon_B-\frac{I^{(0)}_2}{R}\varepsilon_M-\frac{\partial}{\partial x}\left[\mathcal{D}^{(0)}_{21}\right] \tag{20b}$$

$$\mathcal{D}^{(0)}_{30}=-\frac{Q^{(0)}_Y}{2}\varepsilon_N-I^{(0)}_4\varepsilon_M-\left(I^{(0)}_4-N^{(0)}_X\right)\varepsilon_Q-\frac{\partial}{\partial x}\left[\mathcal{D}^{(0)}_{31}\right] \tag{20c}$$

$$\mathcal{D}^{(0)}_{11}=N^{(0)}_X\varepsilon_N-M^{(0)}_Z\varepsilon_M+\left(\frac{I^{(0)}_4}{R}-\frac{Q^{(0)}_Y}{2}\right)\varepsilon_B \tag{20d}$$

$$\mathcal{D}^{(0)}_{21}=N^{(0)}_X\varepsilon_B+I^{(0)}_1\varepsilon_Q \tag{20e}$$

$$\mathcal{D}^{(0)}_{31}=I^{(0)}_3\varepsilon_M-I^{(0)}_2\varepsilon_N-I^{(0)}_4\varepsilon_B \tag{20f}$$

with

$$\left\{N^{(0)}_X,M^{(0)}_Z,Q^{(0)}_Y\right\}=\int_A\left\{\sigma^{(0)}_{xx},y\sigma^{(0)}_{xx},\sigma^{(0)}_{xy}\right\}dA \tag{21a}$$

$$\left\{I^{(0)}_1,I^{(0)}_2,I^{(0)}_3\right\}=\int_A\sigma^{(0)}_{xx}\left\{1,y,y^2\right\}\mathcal{F}dA,\; I^{(0)}_4=\int_A\sigma^{(0)}_{xy}ydA \tag{21b}$$

whereas $\mathcal{M}_j$, $\mathcal{P}_j$ can be easily obtained from (15) and (17a).

2.4 Constitutive Equations

The constitutive equations of the stresses (as well as stress resultants) in terms of the generalized deformations can be obtained consistently by means of Eq. (2b). As a first step, it is useful to note that, according to Eq (1), the elasticity moduli can be expressed as:

$$\begin{aligned}E(y)&=E^+\overline{E}(y)\\ G(y)&=G^+\overline{G}(y)\end{aligned} \tag{22a-b}$$

That is, the material properties can be represented by a product of a dimensional material property factor and a function which is independent of dimensional concerns. Now the normal and shear stresses can be represented, in terms of the forces or beams stress resultants, by means of the following expressions:

$$\sigma_{xx}(y) = \overline{E}(y)\frac{\mathcal{F}}{\Delta^*}\left[N_X\left(J_{22}^* - yJ_{12}^*\right) + M_Z\left(yJ_{11}^* - J_{12}^*\right)\right] \tag{23a}$$

$$\sigma_{xy}(y) = \left[\frac{\mathcal{F}^2\overline{\lambda}_y(y)}{\Delta^*}\right]Q_Y \tag{23b}$$

where:

$$\Delta^* = J_{22}^* J_{11}^* - \left(J_{12}^*\right)^2 \tag{24a}$$

$$\overline{\lambda}_y(y) = -\int_{-h/2}^{y}\overline{E}(y)\left[y\left(J_{11}^* + \frac{J_{12}^*}{R}\right) - \left(J_{12}^* + \frac{J_{22}^*}{R}\right)\right]dy \tag{24b}$$

with:

$$J_{ij}^* = \int_A \overline{E}(y)\left(\overline{g}_i \cdot \overline{g}_j\right)\mathcal{F}dA\,, \quad \text{with} \quad \overline{g} = \{1, y\} \tag{25}$$

The stresses described in Eq. (23) verify the following differential equation (represented in cylindrical co-ordinates) for static equilibrium without presence of volume forces:

$$\frac{\partial\sigma_{xy}}{\partial y} + \frac{2\mathcal{F}}{R}\sigma_{xy} + \mathcal{F}\frac{\partial\sigma_{xx}}{\partial x} = 0 \tag{26}$$

Thus, substituting Eq. (22) and Eq. (23) into Eq. (2b), performing the common steps of the variational calculus and after a brief algebraic handling, one obtains the following expressions for the beam stress resultants or beam forces, as follows:

$$\begin{Bmatrix} N_X \\ M_Z \end{Bmatrix} = E^+\begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{12}^* & J_{22}^* \end{bmatrix}\begin{Bmatrix} \varepsilon_N \\ -\varepsilon_M \end{Bmatrix} \tag{27a}$$

$$Q_Y = G^+\Delta^*\frac{\int_y\left(\overline{\lambda}_y\mathcal{F}^2\right)dy}{\int_y\left(\dfrac{\overline{\lambda}_y^2\mathcal{F}^3}{\overline{G}(y)}\right)dy}\varepsilon_Q \tag{27b}$$

The form of Eq. (23b) contains by itself the shear correction factor associated with every first order shear beam theory without imposing that shear factor arbitrarily as it may be seen in many recent papers (Oh et al., 2003; Chakraborty et al., 2003 among others).

It is interesting to mention that the present model collects in a unified formulation the equations to solve curved beams with four different types of materials, i.e. for isotropic material, for ceramic material, for graded material and also for especially orthotropic material as well.

3 SOLUTION OF THE MOTION EQUATIONS

In order to study vibratory patterns of this type of structures, the motion equations are solved with two methodologies: a) a power series solution and b) a finite element approximation.

3.1 A Power Series Solution for the Eigenvalue problem

The exact solution of the eigenvalue problem can be carried out by means of a generalization of the power series scheme developed originally by Filipich et al. (2003) and Rosales and Filipich (2006) for structural problems involving isotropic materials. The methodology requires a previous non-dimensional re-definition of the differential equations, which implies that $\bar{x} = x / L \in [0,1]$, being L the circumferential length of the curved beam.

The displacement variables have the common harmonic motion:

$$\{u_{xc}, u_{yc}, \theta_z\} = \{u_1, u_2, u_3\} e^{i\Omega t} = \bar{u}.e^{i\Omega t} \tag{28}$$

where Ω is the circular frequency measured in rad/seg, t is the temporal variable and $i = \sqrt{-1}$, whereas the generic displacement $u_i(\bar{x})$ is expanded with the following power series:

$$u_i = \sum_{k=0}^{M} \widehat{C}_{ik} \bar{x}^k \ , \quad i = 1,\ldots,3 \tag{29}$$

Theoretically $M \to \infty$, however for practical purposes M may be an arbitrary large integer. Now employing Eq. (29) in the differential equations given by Eq. (18) then the non-dimensional differential equations can be arranged in the following form:

$$\sum_{j=1}^{3} f_{ij}(\bar{x}) u''_j(\bar{x}) + \hat{F}_i(\bar{u}, \bar{u}') + \lambda\, u_i(\bar{x}) = 0\,, \ \ i = 1,\ldots,3 \ \text{ and } \ \bar{x} \in [0,1] \tag{30}$$

The functions $f_{ij}(\bar{x})$ and $\hat{F}_i(\bar{u}, \bar{u}')$ which are considered analytic $\forall\ \bar{x} \in [0,1]$ condense the cross-section geometric properties together with the constitutive equations. The eigenvalue λ is related to the circular frequency by means of the following expression:

$$\lambda = 2\pi\Omega R^2 \sqrt{\frac{\rho^+}{b^2 E^+}} \tag{31}$$

Applying the boundary conditions in non-dimensional form and appealing to a recurrence scheme (Filipich et al, 2003; Piovan et al., 2008) of the power series one can represent the solution system in the following form:

$$\begin{bmatrix} \varepsilon_{11}(\lambda) & \varepsilon_{12}(\lambda) & \varepsilon_{13}(\lambda) \\ \varepsilon_{21}(\lambda) & \varepsilon_{22}(\lambda) & \varepsilon_{23}(\lambda) \\ \varepsilon_{31}(\lambda) & \varepsilon_{32}(\lambda) & \varepsilon_{33}(\lambda) \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

From where one can obtain the common solution for the eigenvalue problem (for further explanations see Piovan et al., 2008; Filipich et al, 2003 and Rosales and Filipich, 2006).

The recurrence scheme (Piovan et al., 2008) allows to shrink the algebraic problem from 3(M+1) unknowns to only 3 unknown coefficients that can be selected according to the boundary equations.

3.2 The Finite Element Formulation

The structural model developed in the second section is numerically implemented with the finite element method. A five-node iso-parametric element is employed to discretize the structure. This element can represent appropriately the mechanics of the “in-plane” motions (Oñate, 1992). The interpolation of the generic displacements can be carried out by means of the following expression:

$$U_i(\bar{x}) = \sum_{j=1}^{5} f_j(\bar{x}) U_i^{(j)} \quad \text{with} \quad i = 1,...,3 \quad \text{with} \quad \bar{x} = \frac{x}{L_e} \in [0,1] \tag{33}$$

where:

$$\begin{gathered} f_1(\bar{x}) = 1 - \frac{25}{3}\bar{x} + \frac{70}{3}\bar{x}^2 - \frac{80}{3}\bar{x}^3 + \frac{32}{3}\bar{x}^4, \; f_2(\bar{x}) = 16\bar{x} - \frac{208}{3}\bar{x}^2 + 96\bar{x}^3 - \frac{128}{3}\bar{x}^4 \\ f_3(\bar{x}) = -12\bar{x} + 76\bar{x}^2 - 128\bar{x}^3 + 64\bar{x}^4, \; f_4(\bar{x}) = \frac{16}{3}\bar{x} - \frac{112}{3}\bar{x}^2 + \frac{224}{3}\bar{x}^3 - \frac{128}{3}\bar{x}^4 \\ f_5(\bar{x}) = -\bar{x} + \frac{22}{3}\bar{x}^2 - 16\bar{x}^3 + \frac{32}{3}\bar{x}^4 \end{gathered} \tag{34}$$

The vector of displacements can be described as:

$$\{\boldsymbol{U}\} = \left\{\{\boldsymbol{U}^{(1)}\}, \{\boldsymbol{U}^{(2)}\}, \{\boldsymbol{U}^{(3)}\}, \{\boldsymbol{U}^{(4)}\}, \{\boldsymbol{U}^{(5)}\}\right\} \tag{35}$$

with

$$\{\boldsymbol{U}^{(i)}\} = \left\{U_1^{(i)}, U_2^{(i)}, U_3^{(i)}\right\}^T = \left\{u_{xc_j}, u_{yc_j}, \theta_{z_j}\right\}^T \tag{36}$$

Now substituting Eq (34) in (33) and then in Eq. (9a) to (9d) and considering a harmonic motion, one obtains the classic equation of the finite element method for eigenvalue problems considering a state of initial stresses:

$$\left([K] + [K_G] - \Omega^2 [M]\right)\{W\} = \{0\} \tag{37}$$

where $[K]$, $[K_G]$ and $[M]$ are the global elastic stiffness matrix, global geometric stiffness matrix and global mass matrix, respectively; whereas $\{W\}$ is the global vector of finite element variables. $\Omega = 2\,\pi\, f$ is the circular frequency, whereas f is the frequency measured in Hertz. The matrix $[K_G]$ is calculated with the initial stresses obtained by means of the finite element equation corresponding to the condition given by Eq. (3). In order to avoid the shear locking phenomenon, reduced integration is employed in terms related to the shear deformability.

4 NUMERICAL STUDIES

4.1 Some Preliminary Comparisons

In this section a few comparisons are performed in order to show the usefulness of the model. The first study corresponds to a convergence test of the finite element approximation which is compared with the exact solution given by the power series methodology. A curved beam with $R = L = 0.1$ m, $2h = b = 0.02$ m. The properties are graded according to Eq. (1c) from steel at the inner radius and ceramic of alumina in the outer radius (see Table 1 for material properties). The power exponent $n = 1.2$. The curved beam is clamped at one end and free in the other. The error in percentage of the first three non-dimensional frequencies is shown in Figure 3. Notice that with only two finite elements, the error reach values lesser than 0.1 % for the three eigenvalues. The exact solution (or with arbitrary precision) has been carried with series of a one hundred terms.

Properties of materials	Steel	Alumina (Al_2O_3)
Young's Modulus E (GPa)	214.00	390.00
Shear modulus G (GPa)	82.20	137.00
Material Density ρ (Kg/m^3)	7800.00	3200.00

Table 1: Properties of metallic and ceramic materials.

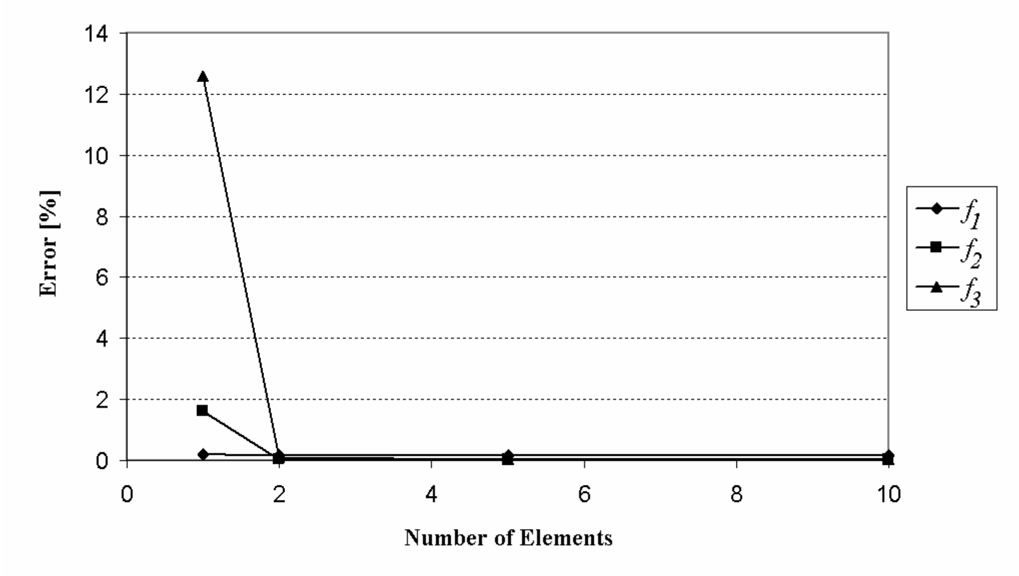

Figure 3: Convergence of the beam element approximation with respect to the exact values.

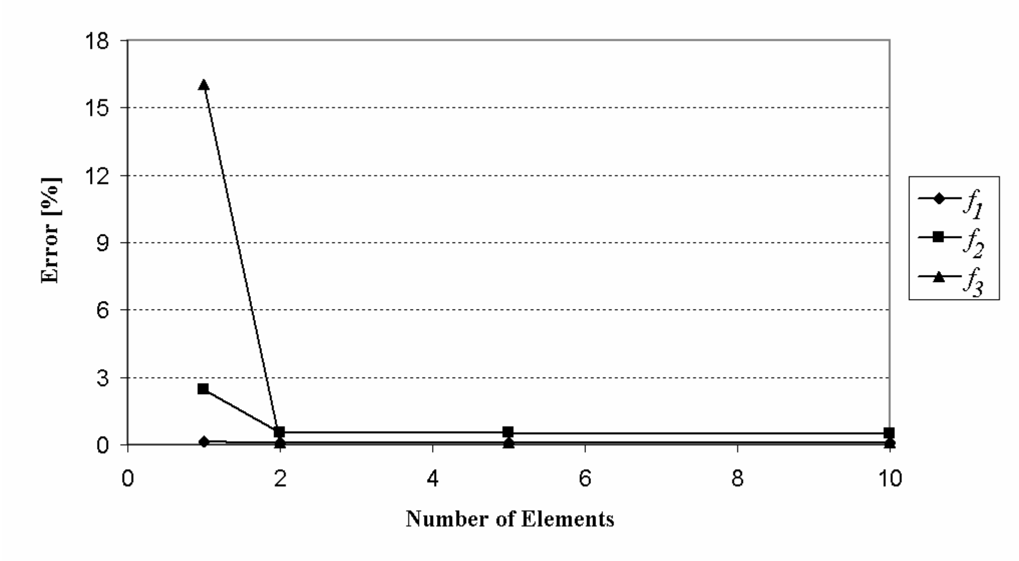

Figure 4: Convergence of the beam element approximation with respect to a 3D finite element approximation.

The second comparative example is based on a curved beam with the same features of the previous example. The beam approach is compared with a 3D finite element approximation performed in a special finite element solver called FlexPDE, where properties can be defined in a formula style (see http://www.pdesolutions.com for further explanations and illustrative examples of the program) that makes it quite useful when complex constitutive equations have to be included in the structural model. In Figure 4 one can see the convergence test, based in the percentage error, of the first three frequencies

4.2 Parametric Studies

The following parametric studies are performed with the finite element approach presented in this paper. The cantilever curved beam has the geometric properties $R = L = 50\ cm$, $b = 2\ cm$ and $h = 0.005\ m$. The three typical laws given in Eq (1a) to (1c) for graded properties are tested and compared. In Figure 5 one can see the variation of the frequency coefficients of the first three eigenvalues with respect to the exponent n corresponding to the gradation law represented by Eq. (1c) with metallic internal surface (at $R=R_i$) and a ceramic outer surface (at $R=R_e$). In Figure 6, the gradation laws given by Eq. (1c) and Eq (1a) are compared. The gradation properties described by Eq. (1a) imply a metallic core and inner and outer ceramic surfaces (at both $R=R_e$ and $R=R_i$).

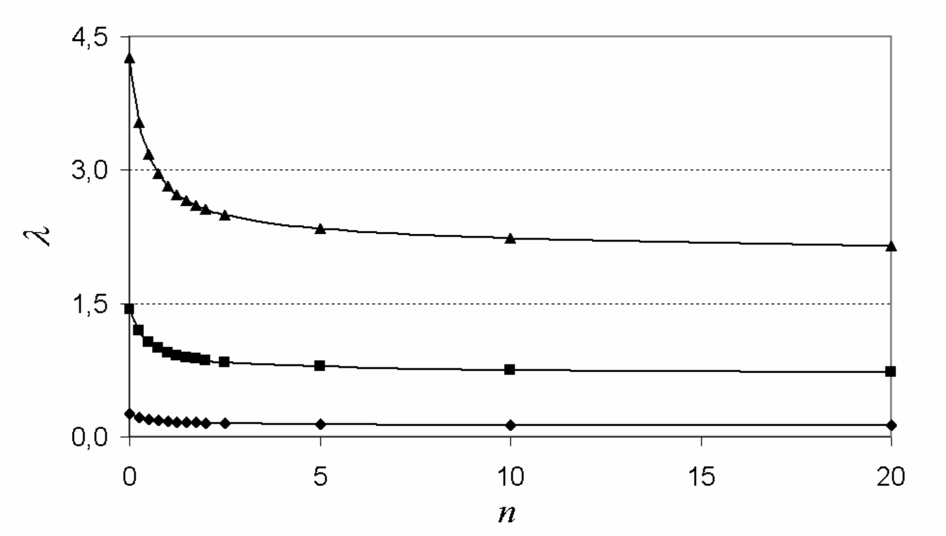

Figure 5: Variation of the first three frequencies for a metallic/ceramic beam with respect to the exponent n. (◆) First frequency coefficient, (■) second frequency coefficient, (▲) third frequency coefficient.

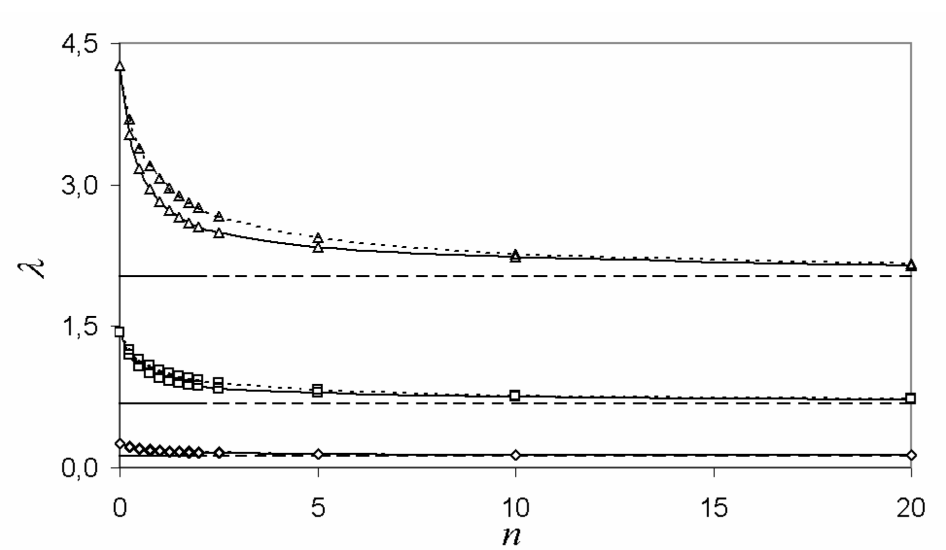

Figure 6: Variation of the first three frequencies with respect to the exponent n. Comparison of different laws for graded properties: (--) Metallic: Steel, (—) Ceramic/Metallic, (····) Metallic core/Ceramic surfaces.

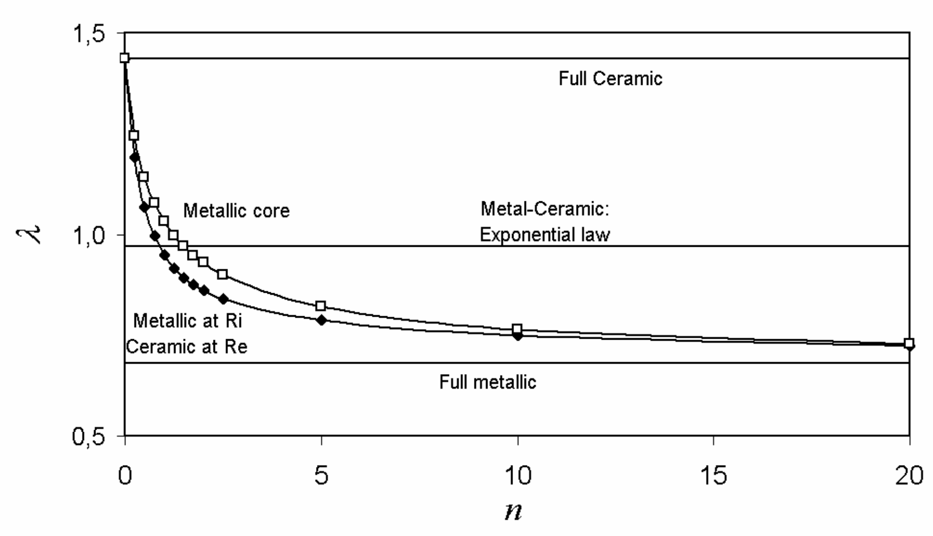

Figure 7: Variation of the frequencies with the exponent *n*. Detail of the second frequency

In Figure 7 one can see a detail of the variation of the second frequency coefficient with the exponent *n*. Note that the exponential law for graded properties, i.e. Eq (1b) does not vary with the mentioned exponent, in fact a beam constructed with this material law can have only a frequency for a defined metallic/ceramic ratio. This law also offers the possibility to represent as limiting possibilities the full metallic case and the full ceramic cases. It is interesting to note that when n = 20, for both laws given by Eq. (1a) and (1c) the beam is composed in more than 95% by metallic phase.

5 CONCLUSIONS

In this paper a new model considering shear deformability in curved beams made of functionally graded materials has been introduced. The model incorporates in a unified fashion the motion equations of the curved beam as well as consistent derivation of the constitutive equations due to shear deformability hypotheses. The model can be employed with different types of gradation laws, and it also contains as particular cases the use of ceramic materials or metallic materials and even especially orthotropic materials. This model can offer certain elementary solutions for curved beams of graded materials as a part of more general modeling of non-homogeneous curved beams; however this is part of current research.

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