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THE DISCRETE NON LOCAL (DNL) RADIATION BOUNDARY CONDITION

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ABSTRACT

A general methodology for developing absorbing boundary conditions is presented [1,10,11]. In the plane case, it is based on a straightforward solution of the system of ODE's that arise from partial discretization in the directions transversal to the artificial boundary. This leads to an eigenvalue problem of the size of the number of degrees of freedom in the lateral discretization. The eigenvalues are classified as in-going or right-going and the absorbing boundary condition consists in imposing a null value for the in-going modes, leaving free the right-going ones. Whereas the classification is straightforward for operators with definite sign, like the Laplace operator, a *"virtual dissipative"* mechanism has to be added in the mixed case, usually associated with wave propagation phenomena, like the Helmholtz equation, or potential flow with free-surface (the *"wave resistance problem"*). Numerical examples are presented in two companion papers at this same conference. [2,8].

1. INTRODUCTION

When solving PDE's on unbounded domains by *"in volume"* discretization methods like finite elements or finite differences, a fictitious boundary has to be introduced somewhere in order to get a bounded computational domain. For elliptic operators of definite sign, like the Laplace operator, enforcing a Dirichlet or Neumann boundary condition on this artificial boundary leads to a well posed problem that converges to the unbounded domain problem. For mixed operators, generally associated with wave phenomena, like the Helmholtz equation or potential flow with a free surface (the wave-resistance problem), the limit process of pushing this artificial boundary to infinity may not converge, and some sort of absorbing boundary condition has to be imposed on the artificial boundary.

Absorbing boundary conditions are usually devised by fitting the solution on the outer boundary with some series expansion of the outer solution. This expansion depends on the operator, coordinate system (i.e. whether it is plane or circumferential, cylindrical or spherical, geometry), dimension (2D or 3D), the shape of the outer boundary (i.e.

whether it is the whole circumference or an angular sector in 2D, the whole sphere or a solid angle sector in 3D). In some circumstances the outer solution is not known and this strategy is not available.

In this work we propose a method to find absorbing boundary conditions in a purely algebraic formulation. Given the matrix of coefficients for the problem on a mesh with a structured portion (only several layers of nodes are needed) at the outer boundary, the problem leads to a system of difference equations with constant matrix coefficients. This system can be solved by standard operational methods via an eigenvalue decomposition. The absorbing boundary condition consists in retaining only the in-going modes. Thus, the method can be implemented as a *"black box"* numerical routine that takes as input the matrix of coefficients of the discrete system, provided that the mesh is structured near the outer boundary. In practice a few structured layers are enough.

2. THE HELMHOLTZ EQUATION

2.1. Partial Discretization

Let's take as an example the Helmholtz equation on an infinite strip $|y| < L$ in the (x, y) 2D plane:

$$\begin{aligned} &\Delta\phi + k^2\phi = f \\ &\phi = 0, \quad \text{at } y = \pm L \\ &\text{radiation b.c.'s at } x = \pm\infty \end{aligned} \tag{1}$$

The forcing term f is supposed to have compact support, i.e.:

$$f \equiv 0, \quad \text{for} \quad |x| > L_x \tag{2}$$

The radiation boundary condition will not be specified yet but will be of the form $\{(\partial\phi/\partial x)\} = \mathcal{F}\{\phi\}$, where $\mathcal{F}$ is a (maybe) non local operator. Now assume that a partial discretization by finite elements in the y direction is performed, so that we assume a series of nodes (not necessarily equally spaced) in the y direction so that:

$$\phi(x, y) \sim \hat{\phi}(x, y) = \sum_{k=1}^{N_{\text{slab}}} \phi_k(x)\, N_k(y) \tag{3}$$

Replacing in (1) and integrating by parts in y we arrive to the following linear system of ODE's:

$$\mathbf{M}\boldsymbol{\phi}_{,xx} + (k^2\mathbf{M} - \mathbf{K})\,\boldsymbol{\phi} = \mathbf{f} \tag{4}$$

where the standard 1D finite element matrices are:

$$\begin{aligned} \boldsymbol{\phi} &= [\,\phi_1 \quad \phi_2 \quad \cdots \quad \phi_{N_{\text{slab}}}\,] \\ M_{jk} &= \int_{y=-L}^{+L} N_j(y)\, N_k(y)\; dy \quad \text{(mass matrix)} \\ K_{jk} &= \int_{y=-L}^{+L} N_{j,y}(y)\, N_{k,y}(y)\; dy \quad \text{(1D Laplace operator matrix)} \\ f_j(x) &= \int_{y=-L}^{+L} N_j(y)\, f(x, y)\; dy \quad \text{(load vector)} \end{aligned} \tag{5}$$

$\mathbf{M}$ and $\mathbf{K}$ are positive definite matrices.

2.2. Unbounded solution. "Viscous modes"

System (4) can be decoupled in a series of scalar independent ODE's if we make the change of variables: $\mathbf{U} = \mathbf{S}^{-1}\boldsymbol{\phi}$, where $\mathbf{S}$ is the solution of the following eigenvalue problem:

$$\tilde{\mathbf{K}}\mathbf{S} = \mathbf{M}\mathbf{S}\boldsymbol{\Lambda} \tag{6}$$

where $\tilde{\mathbf{K}} = k^2\mathbf{M} - \mathbf{K}$, $\boldsymbol{\Lambda}$ is a diagonal matrix and $\mathbf{S}$ a nonsingular change of basis matrix. It can be shown from the properties of K and M, that such a decomposition is possible with $\mathbf{S}$ and $\boldsymbol{\Lambda}$ real. Now, due to the minus sign in the definition of $\tilde{\mathbf{K}}$ above, there are a eigenvalues of both signs, let us say:

$$\begin{cases} \lambda_k < 0; & \text{for } 1 \le k \le N_{\text{inv}} \text{ ("inviscide" modes)} \\ \lambda_k > 0; & \text{for } N_{\text{inv}} + 1 \le k \le N_{\text{slab}} \text{ ("viscous" modes)} \end{cases} \tag{7}$$

As usual the eigenvectors are orthogonal with respect to both $\tilde{\mathbf{K}}$ and $\mathbf{M}$:

$$\boldsymbol{\phi}_k^T \cdot \mathbf{M}\boldsymbol{\phi}_j = 0, \quad \boldsymbol{\phi}_k^T \cdot \tilde{\mathbf{K}}\boldsymbol{\phi}_j = 0, \quad \text{if } j \ne k \tag{8}$$

The resulting equation for each component U_k is:

$$U_{k,xx} - \lambda_k U_k = 0 \tag{9}$$

The general solution for $x < -L_x$ (no source term) is:

$$U_k^{\text{left}}(x) = \begin{cases} b_k^{\text{left}+} e^{+i\mu_k x} + b_k^{\text{left}-} e^{-i\mu_k x} & ; \text{for } 1 \le k \le N_{\text{inv}} \\ a_k^{\text{left}+}\, e^{+\mu_k x} + a_k^{\text{left}-}\, e^{-\mu_k x} & ; \text{for } N_{\text{inv}} + 1 \le k \le N_{\text{slab}} \end{cases} \tag{10}$$

where $\mu_k = \sqrt{|\lambda_k|}$. A similar expression is valid for $x > L_x$ but with other coefficients $a_k^{\text{right}\pm}$, $b_k^{\text{right}\pm}$. In order to have a bounded solution for $x \to -\infty$ (resp. $x \to +\infty$) we should have $a_k^{\text{left}-} = 0$ (resp. $a_k^{\text{right}+} = 0$), and then:

$$\left.\begin{aligned} U_{k,x} + \mu_k U_k = 0, \quad & \text{en } x = L^* \\ U_{k,x} - \mu_k U_k = 0, \quad & \text{en } x = -L^* \end{aligned}\right\} \quad k = N_{\text{inv}} + 1, \ldots, N_{\text{slab}} \tag{11.a,b}$$

are the appropriate radiation boundary conditions, and $L^* > L_x$.

2.3. Inviscid modes

The same reasoning can not be applied directly in the case $k \le N_{\text{inv}}$ (inviscid modes) since in this case both $e^{+i\mu_k x}$, $e^{-i\mu_k x}$ do not decay or grow for $x \to \pm\infty$. Now the governing equation for the j-th mode (9) is equivalent to a 1D Helmholtz equation with wave number $k_j^2 = -\lambda_j$. If we impose a given boundary condition (say Neumann or Dirichlet) in a boundary at $x = \pm L^*$ and let $L^* \to \infty$ the solution does not converge to anything and it may even become unbounded for certain L^* (those for which the acoustic cavity is such that k_j^2 corresponds to a *"resonant mode"*). In contrast, for the elliptic modes the solution converges uniformly to that one satisfying (11). In this case, the corresponding equation is equivalent to a 1D steady heat transfer equation with Newtonian cooling. The situation for the inviscid case is rather paradoxical, since one expects that if we have some exciting acoustic source in the mid portion of a very

long tube, and increase the length of the tube indefinitely, the resulting pattern will be independent of the position of the tubes or the type of ending (open, rigid, membrane, or whatever else). The same occurs in 2D or 3D. If we think at an emitting source in the center of a very large rigid sphere ($(\partial\phi/\partial n) = 0$), and let the radius of the sphere go to infinity, the solution does not converge neither, even if in this case the solution decays with some power of the distance to the origin. The solution to this paradox is that in real world *there is always some amount of physical dissipation.*

It is well known that a small dissipation can be added to the Helmholtz equation by replacing k^2 by $k^2 + i\delta$ in (1.a) , with δ a small positive number (also known as a *"Rayleigh viscosity parameter"*). In this case the inviscid λ_k's shift slightly towards the negative imaginary part semi-plane, i.e. $\lambda_k = -a - i\delta'$, with a, δ' real and positive, and $\delta' \to 0$ with δ. The corresponding solutions are now $e^{+i\mu_{k\delta}x}$, $e^{-i\mu_{k\delta}x}$, with ${\mu_{k\delta}}^2 = |\lambda| + i\delta$, and $\mathrm{Re}\{\mu_{k\delta}\} > 0$. The first decays towards $+\infty$ and we call it *"right-going"* and, conversely, the second decays towards $x = -\infty$ ("left-going"). Boundedness of the solution imposes then: $b_k^{\text{left}+} = 0$, $b_k^{\text{right}-} = 0$. We can now take the limit $\delta \to 0$ and the appropriate absorbing boundary condition is:

$$\begin{aligned} U_{k,x} - i\mu_k U_k &= 0 \quad \text{at } x = L \\ U_{k,x} + i\mu_k U_k &= 0 \quad \text{at } x = -L \end{aligned} \tag{12}$$

The classification as left- or right-going comes from the fact that, if we come back from the frequency domain (Helmholtz equation) to the time domain (wave equation), right-going modes have a positive x-component of *group velocity*, and vice-versa.

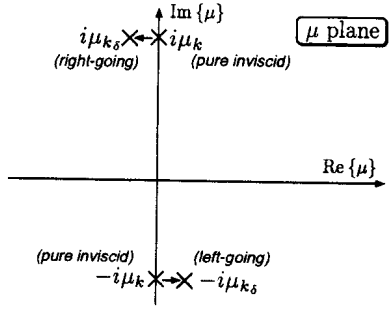

Figure 1: Inviscid eigenvalues are classified by adding a negligible viscosity parameter.

Standard matrix manipulation allows to come back from the $\mathbf{U}$ basis to the $\boldsymbol{\phi}$ basis and the absorbing boundary condition is:

$$\boldsymbol{\phi}_{,x} \mp \mathbf{F} = 0\boldsymbol{\phi}, \quad \text{at } x = \pm L^* \tag{13}$$

where:

$$\mathbf{F} = \mathbf{S}\,\mathrm{diag}\{i\mu_1, \ldots, i\mu_{N_{\text{inv}}}, \mu_{N_{\text{inv}}+1}, \ldots, \mu_{N_{\text{slab}}}\}\,\mathbf{S}^{-1} \tag{14}$$

and $\mathrm{diag}\{a, b, c, \ldots\}$ stands for a diagonal matrix with diagonal entries $a, b, c, \ldots$. The

2.4. Summary of the method

In brief, the method can be described as follows:

- Look for solutions with a dependency $\propto e^{\mu x}$ in the x direction
- Solve the characteristic equation for the eigenvalues $\{\mu\}$. They are classified according to:
  - ▷ Viscous if $\mathrm{Re}\{\mu\} \neq 0$, inviscid otherwise.
  - ▷ The viscous modes are classified as *right-going*, if $\mathrm{Re}\{\mu\} < 0$, *left-going* if $\mathrm{Re}\{\mu\} > 0$.
  - ▷ To classify the inviscid modes, add a small dissipation, and classify them as in the previous point.
- Retain only the right-going modes (viscous or inviscid) in the general expression.
- The radiation boundary condition is found by differentiation of this general form.

Off course, this corresponds to a boundary which is located at the right end of the domain. For a boundary at the left end, the left-going modes should be retained.

It can be shown that the resulting solution is independent of the point where the condition is imposed, provided that this position is outside the region where the source term is non-null or inhomogeneities (k is variable) exists. Several numerical examples on calculations with the Berkhoff's equation for propagation of water-waves in mild-slope bathimetries (very similar to the Helmholtz equation) are shown in [2].

3. FURTHER APPLICATIONS

3.1. The Laplace equation

The Laplace operator (Poisson equation) corresponds to $k = 0$ in the Helmholtz equation. As expected, it can be shown that all modes are "pure viscous". Coming back to the discussion on the first paragraph of §2.3. it is evident that as there are not inviscid modes, one can safely impose Neumann or Dirichlet conditions on the outer boundary, and putting them far enough from the computational domain, the solution will converge to the *"unbounded domain solution"*. Thus, absorbing boundary conditions play a very different role in wave propagation phenomena than for positive definite operators, like the Laplace equation, elasticity, Stokes flow, where all the modes are pure viscous. Whereas in the later case absorbing boundary conditions can be used in order to reduce the size of the computational domain, whenever an inviscid mode exists *an absorbing boundary condition is needed in order to have a convergent solution.* The use of this kind of absorbing boundary conditions is similar to the use of *"infinite elements"* [14].

3.2. The ship wave-resistance problem

When a body moves near the free surface of a fluid, a pattern of trailing gravity waves is formed. The energy spent in building this pattern comes from the work done by the body against the wave resistance. Numerical modeling of this problem is a matter of high interest for ship design, and marine engineering [3-7]. As a first approximation, the wave resistance can be computed with a potential model, whereas for the viscous drag it can be assumed that the position of the surface is held fixed at the reference hydrostatic position, i.e. a plane. This is, basically, the *Froude hypotheses.* With this assumption, we are neglecting the interaction produced by the boundary layer,

which tends to produce a larger body, whose wave pattern, in turn, tends to modify the potential flow which is the input to the boundary layer process. Even if a potential model is assumed for the liquid, the problem is non-linear due to the free surface boundary condition.

Most ship design codes in industry are based on a potential model for the fluid and a linearized free surface boundary condition. The governing equations are the Laplace equation with slip boundary conditions on the hull and channel walls, inlet/outlet conditions at the corresponding planes and the free surface boundary condition. The free surface boundary condition amounts to a Neumann boundary condition with a source term proportional to the streamlined second derivative of the potential. However, the problem as stated so far is ill posed, in the sense that it is invariant under longitudinal coordinate inversion ($x \rightarrow -x$), and it is clear then, that it can not capture the characteristic trailing waves propagating downstream-. To do this, we can either add a dissipative numerical mechanism or impose some kind of "absorbing boundary condition".

It can be shown that the addition of a third order derivative to the free surface boundary conditions, adds a dissipative mechanism and captures the correct sense of propagation for the wave pattern. This is equivalent to use a non-centered discretization scheme for the second order operator and falls among the well known *"upwind-techniques"*. The amount of viscosity added is related to the length of the mesh downstream of the body. If the viscosity parameter is too low, the trailing waves arrive to the downstream boundary, are reflected in the upstream direction and pollute the solution. If it is too high, the trailing waves are damped and incorrect values of the drag are obtained. Extending the mesh in the downstream direction allows the use of a lower viscosity parameter, since the waves are damped in a larger distance, but increases the computational cost (core memory). Numerical experiences show that this third order streamline viscosity term is too dissipative and the meshes should be extended downstream too much. Dawson [4] proposed a method, where the fifth order derivative is used instead, with a very particular finite difference discretization. It is astonishing the fact that standard discretization of the same operator does not work, neither do higher order operators (say seventh order). As a result, most today codes are still using some kind of variant of the Dawson scheme. However, this very particular viscosity term is hard to extend to general boundary fitted meshes, not mentioning to unstructured computational methods like finite elements. It is by this cause that most codes are based on a highly structured panel formulation.

Another possibility suggested here is to use an absorbing boundary condition in the downstream boundary. If such a numerical device could be found, then there is no need to add a numerical viscosity term, since the trailing waves are not reflected upstream, and a usual centered scheme can be used for the free surface boundary term. As a bonus, if such a centered scheme could be used, then the trailing waves would not dampen and the drag could be computed in terms of the momentum flow through a plane arbitrarily located downstream of the body. Absorbing boundary conditions are well studied for other wave phenomena like the Helmholtz equation in acoustics, but are harder to find in the context of the free surface flows studied in this paper. Broeze & Romate [5] developed an absorbing boundary condition for potential flow with a panel method but in the context of following a temporal evolution of the free surface problem.

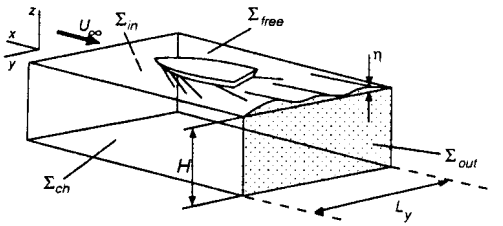

Figure 2: Geometrical description of the ship wave-resistance problem.

Consider the flow around a ship moving at constant speed in a channel of constant section which, for simplicity, is assumed to be a rectangle of depth H and width L_y as shown in figure 2. The fluid to be modeled occupies region Ω which is bounded by: the channel walls and bottom Σ_{ch}, the inlet/outlet boundaries $\Sigma_{\text{in/out}}$, the wetted surface of the ship Σ_{ship} and the free surface Σ_{free}. The governing equations are:

$$\begin{cases} \Delta\phi = 0, & \text{in } \Omega_0 \\ \phi_{,n} = 0, & \text{at } \Sigma_{\text{ch}} \\ \phi_{,n} + K^{-1}\,\phi_{,xx} = -(U_\infty/\rho g)\Delta P_{,x}, & \text{at } \Sigma_{\text{free}} \\ \text{radiation b.c.'s}, & \text{at } \Sigma_{\text{in/out}} \end{cases} \tag{15a-e}$$

Where the ship has been replaced by a pressure distribution ΔP. This is the *"hover-craft problem"*, which is simpler than the full ship problem but retains most of the numerical difficulties associated with the propagation of the gravity waves. Partial discretization of the problem in the transverse direction is performed by assuming an approximation of the form:

$$\phi(x,y,z) \sim \hat{\phi}(x,y,z) = \sum_{k=1}^{N_{\text{slab}}} \hat{\phi}_k(x)\, N_k(y,z) \tag{16}$$

and results in the following system of ODE's:

$$\mathbf{M}\boldsymbol{\phi}_{,xx} - \mathbf{K}\boldsymbol{\phi} + \int_{\Sigma_{\text{free}\,0}} N_j\, \phi_{,n}\, dy = 0 \tag{17}$$

where $\tilde{\mathbf{M}} = \mathbf{M} - K^{-1}\mathbf{M}_{\text{free}}$. $\mathbf{M}$ and $\mathbf{K}$ are as in (5) but integrating over the section of the channel, and $\mathbf{M}_{\text{free}}$ is the *"free surface mass matrix"*:

$$M_{\text{free}.jk} = \int_{\Sigma_{\text{free}\,0}} N_j(y,z)\, N_k(y,z)\, dy \tag{18}$$

and:

$$G_k(x) = (U_\infty/\rho g) \int_{\Sigma_{\text{free}\,0}} N_k\, \Delta P_{,x}\, dy \tag{19}$$

The characteristic equation looks now of the form:

$$\mathbf{K}\mathbf{S} = \tilde{\mathbf{M}}\mathbf{S}\boldsymbol{\Lambda} \tag{20}$$

and due to the negative sign in the definition of $\tilde{\mathbf{M}}$ it results that a certain number of modes (usually equal to the number of nodes on the free surface) are inviscid (see figure 3). As discussed previously, we have to add a dissipative term in order to determine whether they are right- or left-going.

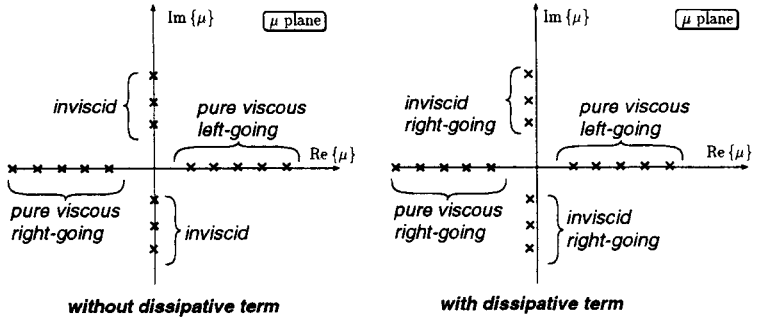

Figure 3: Potential flow with free surface (Wave resistance problem.) Perturbation of the system with a small dissipative term shifts **all** *the inviscid eigenvalues into the right-going plane.*

Other numerical algorithms, notably those based on the work of Dawson, do not use radiation boundary conditions, but instead they add a *numerical viscosity* term proportional to $\delta\,\phi_{,xxxx}$, where δ is a *"numerical viscosity parameter"* (also called a *"Rayleigh viscosity coefficient"*). A term proportional to $-\delta\,\phi_{,xxx}$ is also dissipative but the damping of the waves is too strong to be admissible for numerical calculations. Since after determination of the sense of propagation we take the limit $\delta \to 0$, precision does not matter here, and we choose by simplicity the low order $\phi_{,xxx}$ dissipative term. The perturbed free surface boundary condition is:

$$\phi_{,n} + K^{-1}\,(\phi_{,xx} - \delta\,\phi_{,xxx}) = -(U_\infty/\rho g)\Delta P_{,x} \tag{21}$$

Repeating the semi-discretization process we arrive to the perturbed system of ODE's:

$$\mathbf{M}\,\boldsymbol{\phi}_{,xx} - K^{-1}\mathbf{M}_{\text{free}}(\boldsymbol{\phi}_{,xx} - \delta\,\boldsymbol{\phi}_{,xxx}) - \mathbf{K}\boldsymbol{\phi} = \mathbf{G}(x) \tag{22}$$

A detailed analysis shows that each pair of inviscid eigenvalues differentially move into the $\text{Re}\,\{\mu\} < 0$ semi-plane as shown in the figure, so that all the eigenvalues are right-going. This is in contrast with the Helmholtz case, where for each par of inviscid eigenvalues $\pm i\mu_k$ one of them is right-going and the other left-going. The absorbing boundary conditions are, then, of the form:

$$\begin{cases} U_k = U_{k,x} = 0 & \text{at } x = -L^* \\ \text{none} & \text{at } x = +L* \end{cases} \tag{23}$$

The aspect of the amplitude for both type of modes is shown in figure 4. Note that the inviscid modes have null amplitude upstream of the perturbation region. Once all the modes are summed up and the solution is reconstructed, this results in the characteristic trailing wave pattern following ships.

Further algebraic manipulation leads to expressions similar to (13). However, (23) implies that N_{inv} boundary conditions should be moved from the downstream boundary to the upstream boundary and this causes difficulties for the solution of the linear system. This problem is treated in detail in (11). Numerical results using this technique are shown in a companion paper at this conference [8].

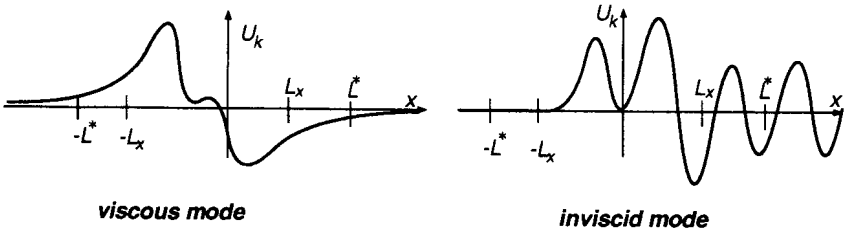

Figure 4: Mode amplitude for the inviscid and pure viscous modes in the wave-resistance problem

3.3. Polar/Spherical coordinates

For the circumferential case (or spherical in 3D), the plane boundary condition is imposed very far from the zone of interest, and then this boundary condition is *"condensed"* through a structured condensation region to the interest zone. The process of condensation can be performed very efficiently in terms of both core memory and CPU time by means of an eigenvalue decomposition which is very similar to that one that leads to the DNL boundary condition. Details of this implementation for the 2D Helmholtz equation are given in [2].

CONCLUSIONS

The DNL methodology allows to develop absorbing boundary conditions for a wide variety of problems in unbounded domains, ranging from positive definite operators (Laplace equations, elasticity equations, for instance) to wave-like propagation phenomena (Helmholtz equation, potential flow with free surface). A great advantage over common methodologies as the DtN from Givoli and Keller is that it can be implemented as a "black box" numerical module that takes as input the matrices relating several layers of nodes on a structured portion of the mesh near the outer boundary. Numerical results are presented in companion papers at this same conference [2,8].

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