Mecánica Computacional Vol. XVII
compilado por G. Etse y B. Luccioni
Tucumán, Septiembre de 1996

NORMAL AXISYMMETRICAL MODES OF OSCILLATION OF A CIRCULAR PLATE WITH A NON-LINEAR FOUNDATION

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Abstract

Several methods have been proposed for constructing the nonlinear normal modes of continuous systems. All of them need the linear eigenfunctions of the problem as a first step. In this paper the case when these eigenfunctions and eigenvalues are not at hand but their approximations, is studied. The method is applied to the calculation of the non-linear axisymmetric frequency coefficients and amplitude-dependent mode shapes of a simply supported circular plate. The plate is partially founded on a cubic non-linear foundation. The approach here presented combines the optimized Rayleigh-Ritz method with the invariant manifold techniques. The approach is applicable to a wide variety of systems. In such cases where the exact solution of the linear system is known the first step may be overridden and the recently developed Shaw and Pierre method may be aplied.

Introduction

Several methods have been recently proposed for constructing the nonlinear normal modes of continuous systems [1-6]. They can be divided roughly into three groups. In the first group it's postulated a dependence of the solution on time in the form:

$$w(x,t) = \sum_{n=0}^{N} \phi_n(x) \cos n\omega t \tag{1}$$

Then the method of harmonic balance is used to obtain nonlinear boundary-value problems of the ϕ_n. In the second group is the Galerkin procedure to discretize the problem. In this case the exact solution is approximated in the form:

$$w(x,t) = \sum_{n=0}^{N} \phi_n(x)\, q_n(t) \tag{2}$$

where $\phi_n(x)$ are the linear undamped mode shapes. Substituting this expansion into both the partial differential equation and the boundary-value conditions a system of ordinary nonlinear equations governing the modal amplitudes $q_n(t)$ is obtained. Then a number of perturbation techniques may be applied to the discretized equations.

Finally the third group, includes the method of multiple scales. The governing partial differential or integral partial differential equations and boundary conditions are directly attacked and no assumptions are made a priori regarding the spatial or temporal dynamics of the system.

It's important to notice that all the above mentioned methods need the eigenfunctions of the linear problem as a first step. In this paper the case when these eigenfunctions and eigenvalues are not at hand but their approximations, is studied. Among a wide variety of methods known as "weighted residuals" both Galerkin and Raleigh-Ritz have proven to be very efficient ones for obtaining solutions of the linear partial differential equations modeling small amplitude vibrations. They work fine in systems with a great variety of boundary conditions [7] and there is a total equivalence between both, in those cases where a functional may be found [8].

Any convenient family of functions satisfying the boundary conditions may be employed as a basis for both the Galerkin projection or the Rayleigh-Ritz functional minimization approach.

Following Shaw [9] normal modes are viewed as motions on invariant manifolds which are tangent to, and of the same dimension as, the linear eigenspaces in the system phase space. These manifolds are the standard foliation of the center-stable manifold of the equilibrium point of interest. The existence and smoothness of these manifolds in the conservative, Hamiltonian case has been extensively studied. One of the important theorems is Lyapunov center's one which states that if the Hamiltonian is C^1 and the linearized system has purely imaginary eigenvalues Ω_j $(j = 1,2,\ldots,m)$ such that for $j \neq k$, $\Omega_j/\Omega_k \neq n$ $(n = 1,2,\ldots)$ then there exists m two-dimensional, local C^1 invariant manifolds. These are the nonlinear normal mode manifolds of interest here, and each one contains a one parameter family of periodic solutions. In what follows this nonresonant condition is supposed to be satisfied. This invariant manifold approach is applied in order to determine how the nonlinear coupling distorts the mode shapes and the dynamics of the nonlinear system. At this point it is important to treat the linear modal amplitudes and velocities as independent variables, as they provide the coordinates which parameterize the manifolds. This procedure provides a systematic mean for determining the amplitude-dependent shapes of natural vibration of weakly nonlinear, distributed parameter systems. The existence of these manifolds is extensively studied in dynamical systems theory [10], but they have not been previously exploited for the construction of nonlinear normal mode shapes and modal dynamics.

As an example we present here the case of a simply supported circular plate partially embedded in a non-linear elastic foundation.

1. The system under study

Systems to which the method is applicable are those described by equations of motion of the form:

$$\frac{\partial^2 w(s,t)}{\partial t^2} + L[w(s,t)] + N[w(s,t)] = 0 \qquad (3)$$

with linear boundary conditions at s = 0, 1

$$B_1[w(s,t)]|_{s=0} = 0 \qquad B_2[w(s,t)]|_{s=1} = 0 \qquad (4)$$

The operator L is taken to be a linear one while the non-linear operator N is assumed to posses smoothness properties that allow it to be expanded to any desired order. Both operators are supposed to act on the spatial variable s only. The case $N = 0$ corresponds to the linear system; the exact solution of this linear problem may be expressed by a standard variable separation in the form:

$$w_0(s,t) = \chi(s)\, exp(i\Omega t) \qquad (5)$$

where the underscript 0 means that we are in the $N \equiv 0$ case
Then a Sturm Liouville problem for the spatial part $\chi(s)$ arises. Any convenient complete family of orthogonal functions $\{\varphi_j(s),\ j=1,2,\ldots,\infty\}$ satisfying the boundary conditions is to be used as a basis for the expansion of $\chi(s)$ as follows:

$$\chi(s) = \sum_{j=1}^{\infty} A_j\, \varphi_j \qquad (6)$$

where the coefficients A_j are obtained by an inner product of the last equation with each basis function. As the basis is an orthogonal set, the A_j's takes the form:

$$A_j = \langle \chi(s), \varphi_j(s) \rangle \qquad j=1,2,\ldots,\infty \tag{7}$$

with the inner product defined as usual:

$$\langle f(s), g(s) \rangle = \int_0^1 f(\xi) g(\xi) d\xi \tag{8}$$

In the particular case of a circular plate with a non-linear elastic foundation here studied, the differential equation is (see Figure 1):

$$D \nabla^4 w + \rho h \frac{\partial^2 W}{\partial t^2} + k_f(r)\, w + \eta_f(r)\, w^3 = 0 \tag{9}$$

where k_f and η_f are constants for $\alpha_1 < r < \alpha_2$ and 0 outside this range and as usual:

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

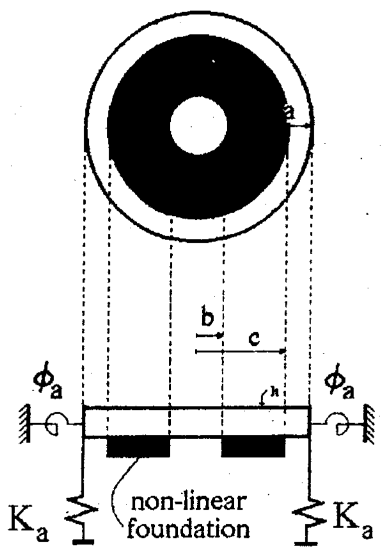

Fig. 1: System under study

2. The First Step: approximating the linear eigenfunctions.

In the Galerkin approach the spatial part of the linear solution is substituted by an approximated one:

$$\chi(s) \cong \chi_a(s) = \sum_{j=1}^{N} A_j\, \varphi_j(s) \tag{10}$$

and then the time dependent approximate linear solution is proposed to be:

$$w_{0a}(s,t) = \chi_a(s)\ exp(\pm i\Omega t) \tag{11}$$

Of course the finite set $\{\varphi_j(s),\ _{j=1,2,\ldots,N}\}$ generates a finite dimensional space instead of the infinite dimensional one, required for the exact representation of $\chi(s)$. Then when w_{0a} is introduced into the differential equation a spatial error function $\epsilon_1(s)$ arises:

$$-\Omega^2\ \chi_a(s) + L[\chi_a(s)] = \epsilon_1(s) \tag{12}$$

The Galerkin condition states:

$$<\epsilon_1(s),\varphi_j(s)> =0 \qquad _{j=0,1,\ldots,N} \tag{13}$$

it means that the error $\epsilon_1(s)$ has no component on the finite dimensional space generated by $\{\varphi_j(s)\}$. This condition implies that the A_j's must satisfy the following system of homogeneous linear equations:

$$-\Omega^2<\chi_a,\varphi_j> + <L[\chi_a],\varphi_j> = 0 \qquad _{j=0,1,\ldots,N} \tag{14}$$

The sets $\{\Omega_i,\ _{i=1,\ldots N}\}$ and $\{A_j^{(i)},\ _{i,j=1,\ldots N}\}$ constitute respectively the N eigenvalues and N eigenvectors of the operator L restricted to the N dimensional space. Once this system is solved the corresponding set of approximate linear modes $\{\chi_a^{(1)},\ldots,\chi_a^{(N)}\}$ is easily obtained. The functions in this set are now to be used as a basis for the second Galerkin projection as is explained below, in the second step description.

On the other hand, in the Raleigh-Ritz method a convenient functional is minimized on the space generated by a selected family of functions satisfying the boundary conditions. If the same trial functions are used the calculations are identical to the Galerkin projection.
In the case here studied an approximation for the plate transverse displacement, convenient in the case of axisymmetric modes of vibration is:

$$W(\bar{r}) \cong W_a(\bar{r}) = \sum_{j=0}^{N}\sum_{i=1}^{3} A_j\ C_{ij}\ R^{\gamma_i+2j} \tag{15}$$

$$C_{ij} = \{1,\ \alpha_j,\ \beta_j\} \quad k = 0,1,2,\ldots \quad \text{and} \quad \gamma_i = \{0,2,4\} \tag{16}$$

where the α_j's and the β_j's are determined by substituting each coordinate function into the governing boundary conditions.

If use is made of the dimensionless variable $r = \bar{r}/a$ and considering the case of a simply supported plate, these boundary conditions can be written in the form:

$$\Phi_a \frac{\partial W}{\partial r}\bigg|_{r=1} = -\left[\frac{\partial^2 W}{\partial r^2} + \mu\frac{1}{r}\frac{\partial W}{\partial r}\right]_{r=1} \tag{17a}$$

$$K_a\ W\bigg|_{r=1} = +\left[\frac{\partial(\Delta W)}{\partial r}\right]_{r=1} \tag{17b}$$

where Δ is the Laplace operator, Φ_a is the adimensional flexibility coefficient for the rotational boundary spring and K_a is the adimensional translational spring constant, given by:

$$\Phi_a = \frac{a}{\phi_a D} \qquad K_a = \frac{k_a\ a^3}{D} \tag{18}$$

Substituting (15) in (17) one obtains:

$$\sum_{i=1}^{3} C_{ij} \left[(\gamma_i+2j)\, \Phi_a + (\gamma_i+2j)(\gamma_i+2j-1) + \mu(\gamma_i+2j) \right] = 0 \qquad (19a)$$

$$\sum_{i=1}^{3} C_{ij} \left[K_a - (\gamma_i+2j)^2(\gamma_i+2j-2) \right] = 0 \qquad (19b)$$

where $j = 0,1,2,\ldots N$.

The appropriate adimensional functional of the problem is:

$$J_{ad}(W) = U_p + U_b + U_f - T_p \qquad (20)$$

where:

$$U_p = \pi D \int_0^1 \left[\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} \right)^2 - 2(1-\mu) \frac{\partial^2 W}{\partial r^2} \frac{1}{r}\frac{\partial W}{\partial r} \right] r \, dr \qquad (21a)$$

$$U_b = \pi D \left[\Phi_a \left(\frac{\partial W}{\partial r} \right)^2_{r=1} + K_a \left(W^2 \right)_{r=1} \right] \qquad (21b)$$

and

$$U_f = \pi D K_f \int_{\eta_1}^{\eta_2} W^2 \, r \, dr \qquad (21c)$$

are the potential energies corresponding to plate strain, plate boundary restraints, and foundation elastic deformation respectively, while

$$T_p = \pi D \, \Omega^2 \left[\int_0^1 W^2 \, r \, dr \qquad (22)$$

is the kinetic energy of the plate and

$$\Omega^2 = \frac{\rho h a^4 \omega^2}{D} ; \; \alpha_1 = \frac{b}{a} ; \; \alpha_2 = \frac{c}{a} ; \; K_f = \frac{k_f a^4}{D}$$

are the frequency coefficient, and the dimensionless radio of inner and outer borders of the foundation respectively.

In accordance with the Ritz method one requires:

$$\frac{\partial J_{ad}[W_a]}{\partial A_j} = 0 \qquad j = 0, 1, \ldots, N \qquad (23)$$

and from the non-triviality conditions one obtains the frequency determinantal equation. It's a relatively easy task to obtain the frequencies with good precision but for the calculation of the modal shapes a naive strategy is not sufficient. The system of linear equations to be solved has a singular coefficients matrix. Pretending that a matrix is either singular or else isn't is of course true analytically. Numerically, however, the far more common situation is that the matrix is numerically very close to singular and roundoff errors in the machine render the equations linearly independent. This problem particularly emerges if the dimension of the matrix is too large. To solve it we employ a Singular Value Decomposition (SVD) technique. SVD methods are based on the following theorem of linear algebra: any m x n matrix **A** whose number of rows m is greater than or equal to its number of columns n, can be written as the product of an m x n column-orthogonal matrix **U**, an n x n diagonal matrix **W** and the transpose of an n x n orthogonal matrix **V**. The matrices **U** and **V** are each orthogonal in the sense that their columns are orthonormal [10].

3. The Second Step: Galerkin projection

The solution of (3) is now approximated by:

$$w_a(s,t) = \sum_{j=1}^{J} \chi_a^{(j)}(s)\; q_j(t) \tag{24}$$

Here $q_j(t)$ represents the contribution of the j[th] linear mode to the response. When substituted into the equation of motion (1) a new error function $\epsilon_2(s)$ is obtained. Repeating the Galerkin procedure above, but now projecting onto each $\chi_a^{(i)}$, the following system of equations arises:

$$\sum_{k=1}^{J} \ddot{q}_k <\chi_a^{(k)},\chi_a^{(i)}> + \sum_{k=1}^{J} q_k <L[\chi_a^{(k)}],\chi_a^{(i)}> + <N\left[\sum_{k=1}^{J} q_k\chi_a^{(k)}\right],\chi_a^{(i)}> = 0 \tag{25}$$

$$(i=1,2,\ldots,J)$$

Taking into account:

$$<L[\chi_a^{(i)}],\chi_a^{(i)}> = <\Omega_i^2\chi_a^{(i)},\chi_a^{(i)}> \tag{26}$$

the system (25) may be rewritten:

$$\sum_{k=1}^{J} (\ddot{q}_k+\Omega_k^2\, q_k)\; <\chi_a^{(k)},\chi_a^{(i)}> + G_i(q_1,\ldots,q_J) = 0 \tag{27}$$

$$(i=1,\ldots,J)$$

This system may be expressed in a matrix form:

$$\mathbf{B}\,(\ddot{\mathbf{q}} + \mathbf{A}\cdot\mathbf{q}) + \mathbf{G} = 0 \tag{28}$$

where

$$\mathbf{B}_{ik} = <\chi_a^{(i)},\chi_a^{(k)}>,\quad \mathbf{A}_k = \Omega_k^2 \text{ and } \mathbf{G}_i = <N\left[\sum_{k=1}^{J} q_k\; \chi_a^{(k)}\right],\chi_a^{(i)}> \tag{29}$$

As $\{\chi_a^{(k)}, (k=1,\ldots,J)\}$ is a set of linearly independent vectors, matrix **B** is non singular and it has an inverse $\mathbf{B}^{-1}$. Pre multiplying equation (16) by this inverse matrix one obtains:

$$\ddot{\mathbf{q}} + \mathbf{A}\cdot\mathbf{q} + \mathbf{M} = 0 \tag{30}$$

where:

$$\mathbf{M}(\mathbf{q}) = \mathbf{B}^{-1}\,\mathbf{G}(\mathbf{q}) \tag{31}$$

For example in the case of a cubic non-linear foundation the vector **M(q)** may be expressed in the form:

$$\mathbf{M}_i(\mathbf{q}(t)) = \sum_{l=1}^{J} \sum_{m=1}^{J} \sum_{n=1}^{J} \sum_{j=1}^{J} \mathbf{B}_{ij}^{-1}\, \mu_{jlmn}\, q_l\, q_m\, q_n \qquad (i=1,\dots,J) \tag{32}$$

with coefficients μ_{jlmn} depending on the nonlinearity.

As a proof of the convergence of the method employed we study de case of a clamped circular plate without foundation. In this case we have the exact solution at hand. It's given by:

$$W_0(r) = C\, J_0\left(\sqrt{\Omega}\, r\right) I_1\left(\sqrt{\Omega}\, r\right) + D\, I_0\left(\sqrt{\Omega}\, r\right) J_1\left(\sqrt{\Omega}\, r\right)$$

with J_i and I_i the first kind and modified Bessel functions respectively.
As an example Figure 2a shows the fundamental modal shape for N = 10 and Figure 2b shows the error between approximate and exact solutions. The maximum value of the error is approximately 10^{-9}. Table 1 depicts the convergence of the fundamental frequency as the number of terms increases.

TABLE I

Convergence of the fundamental frequency in the case of a clamped circular plate without foundation

Number of terms	frequency coefficient Ω_0	exact value
2	10.217027669954404701	
3	10.215827743253612425	
4	10.215826231455139350	
5	10.215826229854428675	10.21582620060082

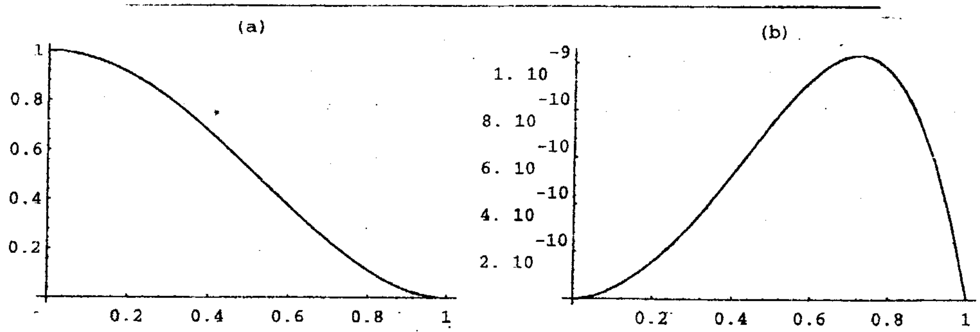

Fig. 2: a) Approximated fundamental modal shape for a clamped circular plate; b) Relative error in the determination of the modal shape in (a)

4. The Third Step: invariant manifold approach

Equation (30) is analogous to that obtained by Shaw. His method may now be applied in a straightforward way as follows: (*i*) equation (30) is written as a first order system form:

$$\dot{q}_i(t) = p_i(t)$$

$$\dot{p}_i(t) = -\Omega_i^2 q_i(t) - M_i(\mathbf{q}(t)) \qquad (i=1,2,\ldots,J) \tag{33}$$

(*ii*) a key observation is to be made [9]: it is that for either a linear or a nonlinear system, a normal mode motion is one in family of motions for which the system behaves like a second-order nonlinear oscillator. Such a motion will take place on a two-dimensional invariant manifold. For weakly nonlinear systems these manifolds are curved but necessarily tangent to the linear eigenspaces at the equilibrium ($w(s,t) \equiv 0$); (*iii*) the construction of the invariant manifolds is carried out in the usual manner by first choosing (q_k,p_k) as natural coordinates which describe the manifold for the k^{th} normal mode and then assuming that the motion on all other (q_i,p_i) pairs can be described in terms of (q_k,p_k); this restricts the dynamics of the entire system to a two-dimensional manifold and defining $(q_k,p_k) = (u_k,v_k)$, a normal mode motion is assumed to exist and be (at least locally) expressible in the form:

$$\begin{aligned} q_i(t) &= Q_{ik}(u_k(t),v_k(t)) \\ p_i(t) &= P_{ik}(u_k(t),v_k(t)) \end{aligned} \qquad i=1,2,\ldots,\infty \tag{34}$$

Substituting (34) in (30) the equations to be satisfied by the Q_{ik}'s and P_{ik}'s are obtained. A closed form solution of these equations is not generally attainable, but a local solution series expansion near the origin can be obtained. The procedure is the standard one for the construction of invariant manifolds [10].

5. The non-linear correction

For this particular case matrices and vectors of (32) take the form:

$$\mathbf{B}_{ij} = 2\pi \sum_{k=0}^{J} \sum_{n=1}^{3} \sum_{l=0}^{J} \sum_{m=1}^{3} \left[A_k^{(i)} A_l^{(j)}\right] C_{nk}\, C_{ml}\, \frac{1}{2k+2l+\gamma_n^{(i)}\gamma_m^{(j)}+1} \tag{35}$$

$$\mu_{iltmn} = \eta_f \left[\sum_{rstu}^{J} A_r^{(i)}\, A_s^{(l)}\, A_t^{(m)}\, A_u^{(n)} \int_{\eta_1}^{\eta_2} \varphi_r\, \varphi_s\, \varphi_t\, \varphi_u\, dx \right] \tag{36}$$

For conservative non-gyroscopic systems, the expansions of $Q_{ik}(u_k,v_k)$ and $P_{ik}(u_k,v_k)$ are restricted to those terms which are consistent with constant amplitude, standing wave normal modes. They are:

$$Q_{ik}(u_k,v_k) = a_{1ik}u_k + a_{3ik}u_k^2 + a_{5ik}v_k^2 + a_{6ik}u_k^3 + a_{8ik}u_k v_k^2 + \ldots \tag{37a}$$

$$P_{ik}(u_k,v_k) = b_{2ik}v_k + b_{4ik}u_k v_k + b_{7ik}u_k^2 v_k + b_{9ik}v_k^3 + \ldots \tag{37b}$$

In this expansion the only non zero coefficients are:

$$a_{1kk} = b_{2kk} = 1 \tag{38a}$$

$$a_{6ik} = \frac{(7\Omega_k^2 - \Omega_i^2)\,\beta_{ikkk}}{(\Omega_k^2 - \Omega_i^2)\,(9\Omega_k^2 - \Omega_i^2)} \tag{38b}$$

$$a_{8ik} = \frac{6\,\beta_{ikkk}}{(\Omega_k^2 - \Omega_i^2)\,(9\Omega_k^2 - \Omega_i^2)} \tag{38c}$$

with

$$\beta_{ilmn} = \sum_{j=1}^{J} \mathbf{B}_{ij}^{-1} \mu_{jlmn}$$

$$b_{7ik} = a_{8ik} \frac{(3\Omega_k^2 - \Omega_i^2)}{2} \tag{39}$$

$$b_{9ik} = a_{8ik} \tag{40}$$

The corrected fundamental frequency is:

$$\Omega_0^* = \Omega_0 \pm \frac{3\ \beta_{0000}\ U_0^2}{8\ \Omega_i} \tag{41}$$

and the corrected fundamental normal mode is:

$$\chi^{(0)*} = U_0\ \chi_a^{(0)} + \sum_{i=2}^{I} \chi_a^{(i)}\ a_{6i0}\ U_0^3 + \ldots \tag{42}$$

6. Conclusions

While the above method is in some respects similar to standard perturbation techniques, a comparison between the results obtained by this method and those obtained using a combination of harmonic balance and eigenfunction expansions shows that different results are obtained for the nonlinear mode shapes, even for the simple example presented here.
The approximations for the frequency of nonlinear modal oscillations agree to first order, but the discrepancies in the mode shape approximations at first order will lead to differences in higher order frequency estimates. The present method is based on the fundamental principle of dynamic invariance and the results obtained here are expected to be the correct ones.
Employing two Galerkin projections is a computationally efficient technique as it allows the use of two different values for the dimension of matrices and vectors: J, a larger one, for the first projection then reducing the approximation error, and N, a lower one, for the second projection then reducing de computing time.

5. Acknowledgements
HAL and DRA want to thank Professor P.A.A.Laura for valuable discussions. This work was partially supported by PID-BID Conicet II 107 and by The University of Mar del Plata (Ing07/94).

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