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NATURAL FREQUENCIES OF NON-HOMOGENEOUS MEMBRANES WITH ARBITRARY INTERFACES

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Abstract. *The study deals with the generalized solution of the title problem. The free vibration problem of a rectangular membrane with partial domains each of uniform density and arbitrary shape is tackled. Previous studies by other researchers include straight parallel to the borders and oblique interfaces, and bent ones. The solution is found by means of a direct variational method with a series composed with a complete set of functions. Two alternative sets are explored: trigonometric and power series. Such series are convergent to the exact solution. The approach is straightforward and very efficient from the computational viewpoint. Diverse illustrations are included. The cases of an oblique straight line interface and an open curve line which divide the membrane in two domains each of different density are first presented. Also a rectangular plate with an interior closed domain is stated and the numerical example of a circular interior zone is included. Comparison between the two alternatives and with other authors' results show excellent agreement. In all cases the results are excellent and the computational cost is very low.*

1 INTRODUCTION

The natural vibration problem of rectangular membranes with arbitrary interface which separates domains with different density is tackled with two alternatives approaches. Two sets of base functions are used in the direct method. They are a trigonometric set and power series. A variational direct approach developed before by the authors and applied to various structural problems that makes use of an extended trigonometric (complete) set (see for instance Rosales,[1] Filipich and Rosales[2] and Escalante *et al.*[3]) is employed to solve the title problem. As is shown the core problem is in someway included in a previous study of vibration of plates (Escalante *et al.*[3]). Also algebraic sets have been previously applied to solve other structural problems (e.g. Filipich *et al.*[4]).

Although many researchers have addressed a great variety of membranes with regions of different density such as rectangular, circular and annular membranes (e.g. references[5–13]), many of them deal with continuous variation of the density or density varying in steps with interfaces parallel to one of the edges. More recently Kang[11] has solved the case of a straight oblique interface and lately the problem of a bent interface.[12] To the authors knowledge no analytical solution is reported in the open literature of a rectangular membrane with regions of different density with curve interfaces, either open or closed.

First the vibrational problem of a rectangular membrane with regions of different density is stated. The direct variational method and the two different approaches are presented. One is a trigonometric set that identically satisfies the boundary conditions of the membrane and constitutes a complete subset and the other is the algebraic set. In the latter the boundary conditions are accounted for by means of Lagrange multipliers. The statement and use of both methodologies is straightforward and the results are of arbitrary precision.

The well-known analogy between the frequency parameter of membranes and simply supported plates is usually employed to solve membranes. This analogy is not valid when the density is not uniform. This theorem has been demonstrated by the author though is not included here for the sake of brevity.

As is known the differential problem of the vibrating membrane is governed by the Helmholtz equation. This also represents the eigenvalue problem of cavities and the results herein presented may be applied to those problems.

2 PROBLEM STATEMENT

The linear natural vibration problem of the membrane (Figure 1) is governed by the following equation (also known as Helmoltz equation):

$$\nabla^2 w + \Omega^{*2} w = 0 \tag{1}$$

under the condition

$$w_{(\Gamma)} = 0 \tag{2}$$

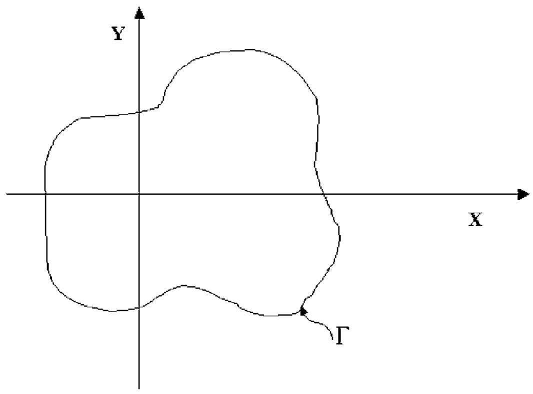

Figure 1: General configuration of a membrane

where

$$\nabla^2(\cdot) \equiv w_{XX} + w_{YY} \tag{3}$$

$$\Omega^{*2} \equiv \frac{\rho(x,y)}{T}\omega^2 \tag{4}$$

in which ω is the natural circular frequency, T is the uniform tension per unit length of edge applied on the membrane and, $\rho(x,y)$ is the variable density per unit of area. The energy functional U^* related with this problem writes

$$2U^* = \iint_{(A)} (w_X^2 + w_y^2 - \Omega^{*2}w^2)dA \tag{5}$$

In the case of a membrane with constant T and variable density on regions (density ρ_i constant in each region i of a total of N regions) (e.g. as shown in Figure 2 for the case $N = 3$), the energy is written as follows

$$2U^* = \int_0^a \int_0^b (w_X^2 + w_y^2)dXdY - \frac{\omega^2}{T}\sum_{n=1}^{N} \rho_n \iint_{(An)} w^2 dXdY \tag{6}$$

It is convenient to non-dimensionalize the problem using the following variables

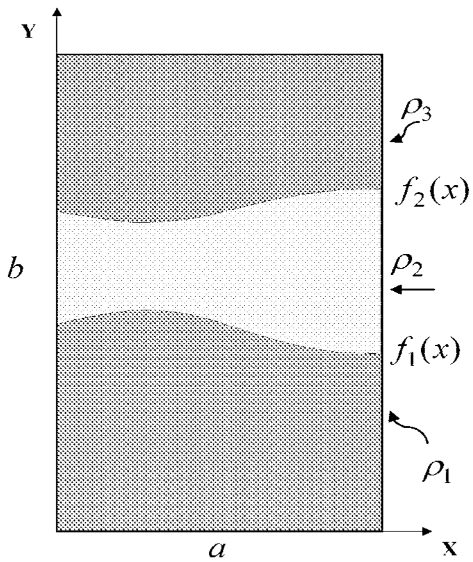

Figure 2: Nonhomogeneous membrane

$$\lambda = \frac{a}{b} \tag{7}$$

$$x \equiv \frac{X}{a};\ y \equiv \frac{\lambda Y}{a} \tag{8}$$

$$r_n \equiv \frac{\rho_n}{\rho_1} \qquad (n = 1, 2, \ldots, N)\ (r_1 = 1) \tag{9}$$

and the functional (6) now writes

$$2U = \int_0^1\int_0^1 (w'^2 + \lambda^2 \bar{w}^2)dxdy - \Omega^2 \sum_{n=1}^{N} r_n \iint_{(An)} w^2 dxdy \tag{10}$$

where the nondimensionalized frequency parameter Ω is

$$\Omega = \sqrt{\frac{\rho_1}{T}}\omega a \tag{11}$$

The arbitrary curves (interfaces) that separate the regions should be given in the form $y = f_m(x)$ with $m = 0, 1, 2, \ldots, N$).

The two alternatives are presented in the following sections.

3 VARIATIONAL SOLUTION WEM

First the title problem is tackled by means of a direct method using an extended trigonometric base set. A complete set is employed to represent the main unknowns with uniform convergence and satisfying the boundary condition $w_{(\Gamma)}$. The authors have developed a methodology named WEM (Whole Element Method) (MEC in Spanish). This direct method (see for instance[1–3]) makes use of a complete series of extended trigonometric functions which are systematically stated for one, two and three domains and also for spatial or time problems, and other complexities. In this work, the vibration of membranes, such series are given by a set that satisfies *a priori* the geometrical boundary conditions and are elementary. Still, they constitute a complete subset. The following complete series are used to represent the variables in the domain $0 \leq x \leq 1$

$$\left\{ \begin{array}{l} w = \sum_{i_1} \sum_{j_1} A_{ij} s_i s_j \\ w' = \sum_{i_1} \sum_{j_1} (i\pi) A_{ij} c_i s_j \\ \bar{w} = \sum_{i_1} \sum_{j_1} (j\pi) A_{ij} s_i c_j \end{array} \right. \tag{12}$$

The notation used in the previous series is as follows: $s_i = sin(i\pi x)$,$s_j = sin(j\pi y)$,$c_i = cos(i\pi x)$,$c_j = cos(j\pi y)$. i_1 denotes $i = 1$ and j_1 denotes $j = 1$. These expansions are introduced in (10) and after finding its extreme, the eigenvalue problem is reduced to the solution of the following equation (the equation is particularized to the case of two regions with densities ρ_1 and ρ_2)

$$\sum_{i_1} \sum_{j_1} \delta A_{ij} \sum_{p_1} \sum_{q_1} A_{pq} \Phi_{ijpq} = 0 \tag{13}$$

with $$\Phi_{ijpq} = \pi^2 ip\epsilon_{ijpq} + \lambda\pi^2 jq\phi_{ijpq} - \Omega^2[\mu^{(1)}_{ijpq} + (\rho_2/\rho_1 - 1)\mu^{(2)}_{ijpq}] \tag{14}$$

$$\epsilon_{ijpq} = \int_0^1 c_i c_p dx \int_0^1 s_j s_q dy \tag{15}$$

$$\phi_{ijpq} = \int_0^1 s_i s_p dx \int_0^1 c_j c_q dy \tag{16}$$

$$\mu^{(1)}_{ijpq} = \int_0^1 s_i s_p dx \int_0^1 s_j s_q dy \tag{17}$$

$$\mu^{(2)}_{ijpq} = \int_\alpha^\beta s_i s_p dx \int_{f_1(x)}^{f_2(x)} s_j s_q dy \tag{18}$$

where α, β and $f_1(x)$, $f_2(x)$ are the limits of the region with different density in the x and y directions respectively.

Numerical examples that illustrate the methodology will be shown in Section Numerical illustrations bellow. They include membranes of two regions of different density with straight or curve interface and a closed inner region.

4 DIRECT METHOD WITH POWER SERIES

A set of algebraic functions (power series) is used as base functions in the direct method. In order to satisfy the boundary condition, Lagrange multipliers are introduced. Instead of using expressions (12) the following power expansions are introduced

$$w = \sum_{i=0} \sum_{j=0} A_{ij} x^i y^j \tag{19}$$

Evidently, this expansion constitutes a complete set in this problem although the boundary conditions of the membrane are not satisfied (as before with functions (12)). Recall that $w(x, y)$ should be null at the boundaries $(\pm 0.5, y)$, and $(x, \pm 0.5)$ if the axes are located in the center of the square membrane (after non dimensionalization). These conditions are to be handled by the use of Lagrange multipliers to expand the governing functional (10). The extreme condition on the functional will lead to the following linear homogeneous system

$$\begin{cases} \mathbf{BA} + \mathbf{K'L} = \mathbf{0} \\ \mathbf{KA} = \mathbf{0} \end{cases} \tag{20}$$

B of components b_{IJ} is a square matrix of order $(M + 1) \cdot (M + 1)$ while **K** of components k_{RS} is a rectangular matrix of order $2(M+N) \cdot [(M+1)(N+1)]$. Also $I \equiv (N+1)i+j+1\ \forall ij$, $J \equiv (N + 1)p + q + 1\ \forall pq$, $(i, p = 0, 1, \ldots, M)$ and $(j, q = 0, 1, \ldots, N)$. The b_{IJ}'s write

$$b_{IJ} = (0.5)^{i+j+p+q} [1 - (-1)^{i+p+1}][1 - (-1)^{j+q-1}] \tag{21}$$

$$\left[\frac{ip}{(i + p - 1)(j + q + 1)} + \lambda^2 \frac{jq}{(i + p + 1)(j + q - 1)} \right] - \Omega^2 \sum_{n=1}^{N^*} r_n \varphi_{ijpq}^{(n)} \tag{22}$$

where

$$\varphi_{ijpq}^{(n)} \equiv \int_{-0.5}^{0.5} x^{i+p} dx \int_{f_{n-1}(x)}^{f_n(x)} y^{j+q} dy \tag{23}$$

and $f_n(x)$ are the limits of the successive regions. Additionally the Lagrange conditions lead to $(S = (N + 1)i + j + 1)$

$$k_{RS} \begin{cases} = (0.5)^i;\ R = j + 1 (i = 0, 1, \ldots, M), (j = 0, 1, \ldots, N - 1) \\ = (-0.5)^i;\ R = j + N (i = 0, 1, \ldots, M), (j = 0, 1, \ldots, N - 2) \\ = (0.5)^i;\ R = i + 2(N - 1) + 1 (i = 0, 1, \ldots, M), (j = 0, 1, \ldots, N) \\ = (-0.5)^i;\ R = i + (M + 1) + 2(N - 1) + 1 (i = 0, 1, \ldots, M), (j = 0, 1, \ldots, N) \end{cases} \tag{24}$$

The eigenvalue problem may be stated as follows if matrix **B** is not singular,

$$| \mathbf{B} || \mathbf{KB'K'} |= \mathbf{0} \tag{25}$$

Table 1: Frequency parameter values of the homogeneous rectangular plate

Order	Exact	Algebraic set			Trigonometric set		
		M=N=4	M=N=9	M=N=11	M=N=4	M=N=8	M=N=10
1	3.5939	3.5939	3.5939	3.5939	3.5939	3.5939	3.5939
2	4.6962	4.7784	4.6962	4.6962	4.6962	4.6962	4.6962
1	6.1062	6.4336	6.1062	6.1062	6.1062	6.1062	6.1062
1	6.5211	6.7116	6.5211	6.5211	6.5211	6.5211	6.5211
1	7.1877	7.4137	7.1877	7.1877	7.1877	7.1877	7.1877
1	7.6556	8.5745	7.6565	7.6556	7.6556	7.6556	7.6556

Table 2: Frequency parameter values of membrane with straight interface. Trigonometric series. Example 1

Order	Frequency parameter				
	M=N=5	M=N=8	M=N=10	M=N=12	Kang and Lee[11]
1	2.7686	2.7681	2.7681	2.7680	2.768
2	3.9686	3.9664	3.9661	3.9659	3.966
3	4.8466	4.8452	4.8450	4.8449	4.845
4	5.0089	5.0062	5.0061	5.0060	5.005
5	5.8535	5.8445	5.8432	5.8427	5.842

Any of the two involved factors may be null but since the eigenvalues obtained with $\mid \mathbf{B} \mid$ do not involve the boundary restrictions, the frequencies are to be found with

$$\mid \mathbf{KB'K'} \mid = \mathbf{0}. \tag{26}$$

5 NUMERICAL ILLUSTRATIONS

First a rectangular membrane of dimensions 1 m in the X direction and 1.8 m in the Y direction is addressed. As a reference the values of a rectangular plate with homogeneous density are reported in Table 1 found with the direct method with two proposed sets. Now the non homogeneous case with two regions of different density ($\rho_2 = 2\rho_1$) separated by an oblique interface is tackled (see Figure 3, Example 1). The interface is given by the line $Y = 0.3X + 0.7$. The values obtained using the Direct Method with trigonometric functions (Eqs.(12)) are depicted in Table 2 for various number of terms of the series. The values reported in Kang and Lee[11] are depicted in the rightmost column. In this reference a superposition of two semi-infinite membranes is proposed and a sum of trigonometric functions employed to solve the governing differential equation. The methodology is somewhat involved and some spurious frequencies are found which are discarded by comparison with other methods (e.g. FEM). On the other hand, the herein presented direct method with a complete trigonometric set yields all the frequencies without being necessary such an analysis of the results.

The case of a square membrane (1 m by 1 m) with a curve interface was also studied. Figure 3, Example2, shows the case of the membrane with two regions with different density and a

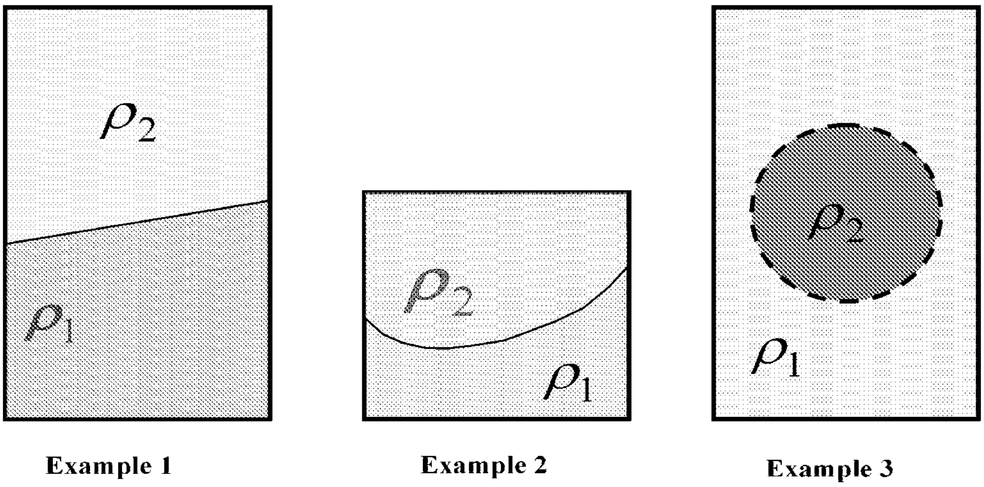

Figure 3: Geometrical scheme of the three examples of nonhomogeneous membranes

curve interface. In particular, the parabola $Y = 0.8X^2 - 0.5X + 0.4$ is chosen as the interface curve. The values of the first five frequency parameters are reported in Table 3, found with the direct method using both the trigonometric and the algebraic sets.

Finally a rectangular membrane (1 m x 1.8 m) with an inner closed region with different density was analyzed and the frequency parameter results depicted in Table 4. In particular, the density of a circular region, with center in the center of the plate and radius 0.4 m is twice, three times and five times the density of the membrane (i.e. $r = \rho_2/\rho_1$, $r = 2, 3, 5$).

As expected, the effect of a region with larger density is to lower the frequencies. Figure 4 shows the influence of the change of density on the sucessive frequency parameters. With respect to the methodology, the direct method offers a very simple and reliable tool to solve this type of problem. Given the simplicity of the boundaries the trigonometric set satisfies the end conditions previously constituting a complete subset. On the other hand the algebraic set (power series) requires the use of Lagrange multiplier to fulfill those conditions. The convergence is better with the trigonometric set and has the particularity of being similar even in the higher frequencies. In the comparison with results from other authors (Example 1), though all methods attain the same values, the advantage of the present methodology is that no "selection" of results have to be performed. Additionally the use of complete sets in the direct method ensures the convergence of the solution to the exact one, i.e. the results are of arbitrary precision. For instance if one needs frequencies accurate to three decimal places, the number of terms should

Table 3: Frequency parameter values of a square membrane with curve interface. Example 2.

Order	Frequency parameter	
	Trigonometric set	Algebraic set
	M=N=10	M=N=8
1	3.3373	3.3374
2	5.2591	5.2595
3	5.6367	5.6377
4	7.0885	7.0904
5	7.4718	7.4729

Table 4: Frequency parameter values of membrane with an inner closed region of different densities. Example 3.

Order	Trigonometric set (M=N=10)			Algebraic set(M=N=11)		
	r=2	r=3	r=5	r=2	r=3	r=5
1	2.7386	2.2856	1.7995	2.7389	2.2860	1.7999
2	4.0156	3.4927	2.8318	4.0168	3.4948	2.8345
3	5.0200	4.1793	3.2807	5.0210	4.1806	3.2821
4	5.4314	4.9317	4.1367	5.4386	4.9614	4.2051
5	6.2559	5.3985	4.3270	6.2600	5.4070	4.3384
6	6.6140	6.1391	4.8361	6.6225	6.1442	4.8414

be increased until the significant digits remain constant. Thus the resulting eigenvalue is exact.

6 FINAL COMMENTS

A direct method approach is herein employed to solve the vibrational problem of non homogeneous rectangular membranes. Two different base functions that belong to complete sets are inserted in the governing functional. Then the variational approach gives the necessary equation to solve the eigenvalue problem. In particular the density is varied in steps and three examples were numerically solved: an oblique interface, a curve interface and an closed inner region. Both the trigonometric and the algebraic sets exhibited an excellent performance, giving between 4 and 5 digits of accuracy with only 10 terms. As mentioned before, the well-known analogy between plates and membranes that is valid with homogeneous membranes, does not hold in the case of non homogeneous membranes. Thus, the frequencies of membranes with different densities may not be found from the plate results.

REFERENCES

[1] M.B. Rosales. *A non-classical variational method and its application to statics and dynamics of structural elements(in Spanish)*. Ph.D. Thesis. Universidad Nacional del Sur, Argentina, (1997).

[2] C.P. Filipich and M.B. Rosales. Arbitrary precision frequencies of a free rectangular thin

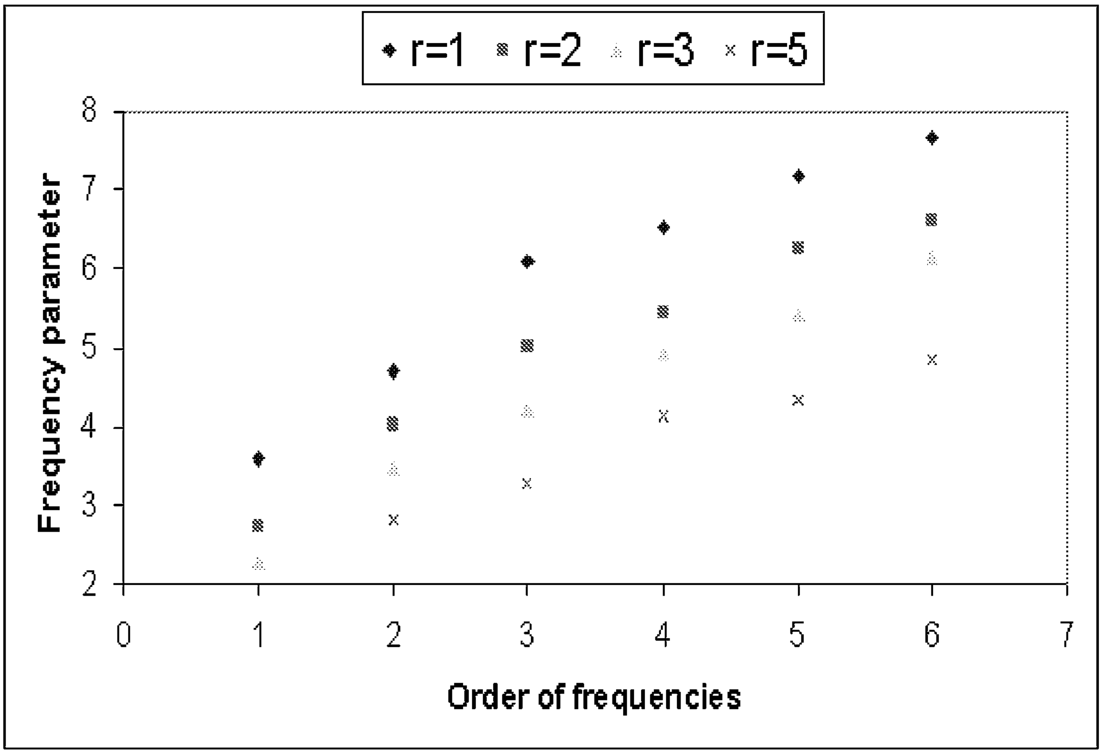

Figure 4: Rectangular membrane with a circular inner region. Influence of density ratio on the frequency parameter

plate. *J. Sound Vibr*, **230**, 521–539 (2000).

[3] M.R. Escalante, M.B. Rosales, and C.P. Filipich. Natural frequencies of thin rectangular plates with partial intermediate supports. *Int. J. Latin Amer. Appl. Res.*, **34**, 217–224 (2004).

[4] C.P. Filipich, M.B. Rosales, and F.S. Buezas. Some nonlinear mechanical problems solved with analytical solutions. *Int. J. Latin Amer. Appl. Res.*, **34**, 105–123 (2004).

[5] K. Sato. Free vibration analysis of composite rectangular membranes consisting of strips. *J. Sound Vibr.*, **49**, 535–540 (1976).

[6] H.P.W. Gottlieb. New type of vibration mode for stepped membranes. *J. Sound Vibr.*, **110**, 395–411 (1986).

[7] V.H. Cortinez and P.A.A. Laura. Vibrations of non-homogeneous rectangular membranes. *J. Sound Vibr.*, **156**, 217225 (1992).

[8] J.A. Masad. Free vibrations of a non-homogeneous rectangular membrane. *J. Sound Vibr.*, **195**, 674–678 (1996).

[9] P.A.A. Laura, R.E. Rossi, and R.H. Gutierrez. The fundamental frequency of non-homogeneous rectangular membranes. *J. Sound Vibr.*, **204**, 373376 (1997).

[10] S.H. Ho and C.K. Chen. Free vibration analysis of non-homogeneous rectangular membranes using a hybrid method. *J. Sound Vibr.*, **233**, 547555 (2000).

[11] S.W. Kang and J.M. Lee. Free vibration analysis of composite rectangular membranes with an oblique interface. *J. Sound Vibr.*, **251**, 505–517 (2002).

[12] S.W. Kang. Free vibration analysis of composite rectangular membranes with a bent interface. *J. Sound Vibr.*, **272**, 39–53 (2004).

[13] G.R. Buchanan. Vibration of circular membranes with linearly varying density along a diameter. *J. Sound Vibr.*, **280**, 407–414 (2005).