FINITE STRIP METHODS FOR INSTABILITY OF PRISMATIC PLATE ASSEMBLIES

Luis A. Godoy
Departamento de Estructuras, F.C.E.F. y N., Universidad Nacional de Córdoba, and CONICET, Argentina.

RESUMEN

El trabajo revisa los fundamentos y aplicaciones del Método de Bandas Finitas en placas plegadas prismáticas. En la primera parte se comenta sobre la motivación de estudiar esos problemas en el contexto de estructuras de ingeniería civil. En la segunda parte se discuten los conceptos energéticos de estabilidad de los que surgen las ecuaciones de equilibrio, y los campos de desplazamientos usados en la aproximación de bandas finitas. La última parte está orientada a aplicaciones de los fundamentos de estabilidad y bandas finitas a la solución de problemas de bifurcación, equilibrio poscrítico e interacción modal. Las aplicaciones muestran que la técnica permite resolver muchos problemas de estabilidad de placas plegadas prismáticas, con las restricciones de las condiciones de contorno que pueden satisfacerse en problemas de pandeo global.

ABSTRACT

The fundamentals and applications of the Finite Strip Method to instability of prismatic plate assemblies are reviewed. In the first part of the paper the motivation to study such stability problems in the context of civil engineering structures is stated. In the second part both the energy concepts of stability from which the equilibrium equations are obtained, and the displacement fields used for the finite strip approximations are discussed. The last part is oriented to applications of the stability and finite strip fundamentals to the solution of bifurcation buckling, post buckling equilibrium and mode interaction. The applications show that the technique is capable of handling almost every problem of stability of prismatic plate assemblies, with the restrictions of the boundary conditions that can be satisfied in global buckling problems.

MOTIVATION OF THE STUDY IN CIVIL ENGINEERING STRUCTURES

Structures composed by assemblies of flat, thin plates (either folded plates or thin walled beams) are often susceptible to buckling. In steel structures, in which each plate is very thin, local buckling may occur in a similar way to buckling of simple supported plates. However, the plate assembly usually has considerable post buckling strength and it may carry loads well over the local buckling load and before actual collapse of the structure occurs. For such class of structures the evaluation of the post buckling response and the prediction of advanced states of deformation are of great interest. Even an elastic analysis (coupled with a plasticity criterion) will render good estimates of maximum loads which the structure may carry in very thin plate situations. For somewhat thicker structures, plasticity effects are important and elasto-plastic constitutive equations should be considered if real collapse needs to be approximated. For such kind of study, the influence of geometric imperfections may considerably affect the response.

In reinforced concrete structures local buckling is unlikely to occur because of the thickness of the plates, and global modes will lead to the lowest buckling loads. Two types of modes may occur in this case: either flexural or torsional modes, depending on the loading system and stiffness of the cross section. A study of the post buckling behaviour of reinforced concrete folded plates and thin walled beams is a very difficult task, but relatively small post buckling strength is to be expected. For such concrete structures, small increments in post buckling load produce large deflections and consequent deterioration of the material in the form of cracks, and this induces the collapse of the whole structure at a load slightly higher than the bifurcation load. Thus, bifurcation buckling (even elastic) seems to be a good estimate of real collapse loads in reinforced concrete plate assemblies.

Mode interaction in the instability process is only likely to occur in steel structures. For certain cross section and length characteristics, coupling between local and overall buckling modes may lead to reduced buckling loads displaying imperfection sensitivity and unstable post buckling behaviour.

Thus, the type of instability study to be carried out depends on the characteristics of the structure (both, geometry and material) and loading conditions. According to that, the following instability studies may be necessary:

1) Bifurcation load, from a linear, elastic fundamental path in a perfect structure. This leads to an eigenvalue problem.

2) Initial post-buckling path, that is, evaluation of the curvature of the post buckling path at the bifurcation point following Koiter's energy analysis.

3) Post buckling path, in which case the non-linear equations have to be solved to obtain equilibrium states beyond the bifurcation point. This is usually done for structures which show a stable post-buckling path in the elastic range.

4) Mode interaction in the post-critical range.

5) Non-linear fundamental path for imperfect structures.

NUMERICAL METHODS FOR STABILITY STUDIES

The two most general techniques for the analysis of structures composed of plate assemblies are the finite difference and the finite element methods in which a two dimensional discretization is made for each constituent plate. As such, they have no limitations regarding boundary conditions which can be satisfied; can also take into account local discontinuities such as openings and transverse stiffeners in thin-walled beams; and need not distinguish between short and long end-supported structures. However, the linear static analysis of thin shells and folded plates using such two-dimensional discretizations requires a large number of degrees of freedom and the assembled system of equations to be solved is often extremely large. And if eigenvalue or non-linear problems, such as those discussed in the previous section, are to be solved, the computational effort may become prohibitive.

For the class of problems of end-supported structures which are continuous between supports, a semi-analytical technique may be used with advantages over a full two-dimensional discretization. The Finite Strip Method (FSM), which falls into the category of semianalytical (finite element) methods, has become very popular since 1970, its development being associated with the name of Cheung since the mid 1960s. It has been applied with success to compute stresses and natural frequencies of plates and plate assemblies; but although the book of Cheung [2] makes almost no reference to instability problems, the FSM has in it one of the most important fields of application because of the economies that may be obtained with respect to the traditional two dimensional discretizations.

The aim of this paper is to give a picture of the state of the art of applications of the FSM to instability problems in structures composed by plate assemblies. The paper will centre on investigations related to critical loads, which is the starting point of most stability studies, but will also briefly review the work done to determine post-buckling behaviour. First, the stability problem is discussed using the total potential energy functional; second, the use of compatible and incompatible finite strip displacement fields is reviewed. Third, applications of the FSM to compute bifurcation loads in both axially loaded and transversely loaded plate assemblies are presented. The last section gives a survey of some important applications of the FSM to determine post buckling behaviour and collapse of the structures under consideration.

FORMULATION OF THE STABILITY PROBLEM

The study of equilibrium states of slender structures may show instability in basically two different ways: either at a bifurcation point or at a limit point. In real structures, bifurcation is not observed because the presence of even small imperfections in geometry and load have the effect of transforming the bifurcation of two different equilibrium paths into a single non-linear path. But still the concept of bifurcation is a very usefull one because it provides an initial measure of the instability process, and in many cases it is closely associated to collapse.

The energy approach to study critical and initial post-critical states in elastic structures was developed by Koiter in 1945 [10] and used by himself and a number of authors in the context of plate and shell structures (see, for example, the review by Tvergaard [18]).

For a thin walled-structure such as the one shown in Fig. 1, in which the global coordinate axis are X_r $(r = 1,3)$ and the local coordinate axis for each plate are x_r, the change in total potential energy π between a state without stresses and deformations, and an arbitrary deformed state may be written as

$$\pi = \sum_{k=1}^{K} \Big\{ \frac{t}{2} \int_{x_1} \int_{x_2} (n_{ij} E_{ij} + m_{ij} X_{ij})\, dx_1\, dx_2$$

$$- t \int_{x_1} \int_{x_2} (p_i u_i + p_3 u_3)\, dx_1\, dx_2 \quad -$$

$$- t \int_{x_2} \big[P_1 U_1 \big]_{x_1=0}^{x_1=1} dx_2 \Big\} \qquad (i,j = 1,2) \qquad (1)$$

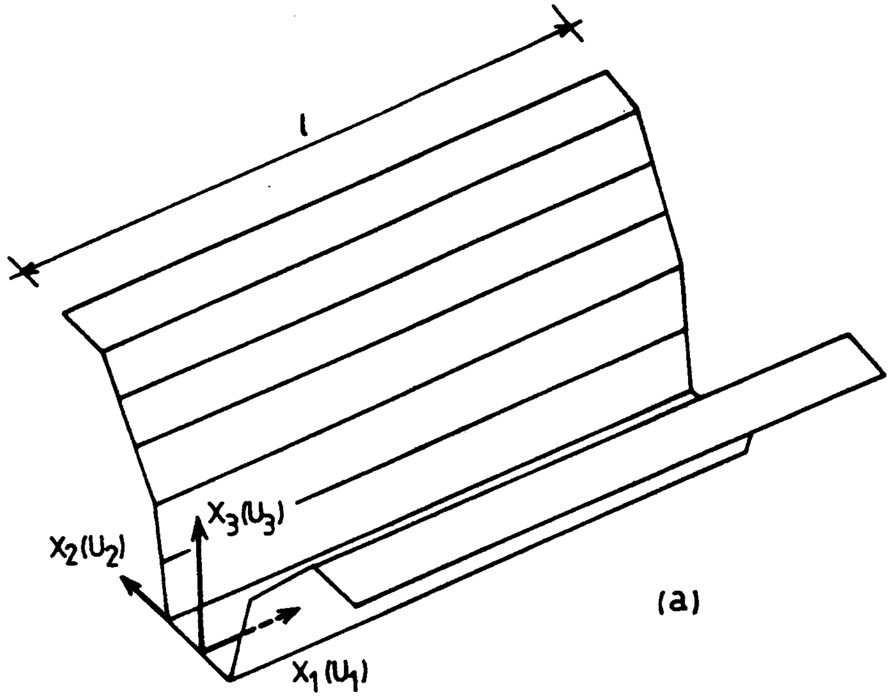
in which n_{ij}, m_{ij} represent the stress and moment resultants in the thickness t of the plates; E_{ij}, X_{ij} are the deformations and changes in curvature; u_i, u_3 the displacement components; p_i, p_3 the distributed load components; and P_1 the axial loads applied at the ends $x_1 = 0$ and $x_1 = 1$ in which the structure is supported. Line loads applied at joints between plates may also be included without difficulties. The strain-displacement and stress-strain relationships that may be used are given in Appendix I.

If bifurcation occurs away from the fundamental equilibrium path into a secondary path, the displacements in the latter can be written in the form

$$u_i^s = u_i^f + u_i$$

$$u_3^s = u_3^f + u_3 \qquad (2)$$

in which $(\ )^s$ represents displacements in the secondary path; $(\ )^f$ refers to the fundamental state, and variables without supraindex are incremental displacements measured from the fundamental state.



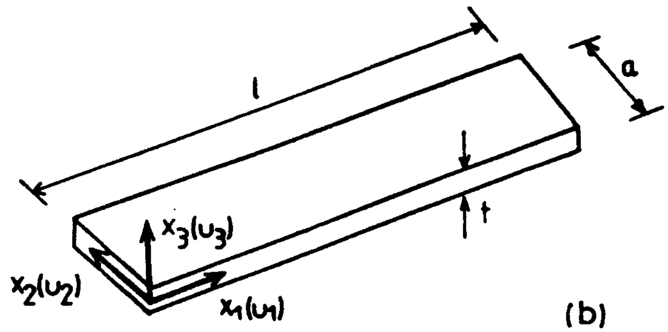


Figure 1 - (a) Coordinate axis for plate assembly;
(b) Finite Strip Configuration

$$\varepsilon^{s}_{ij} = \lambda \; \varepsilon^{f}_{ij} + \varepsilon'_{ij} + \varepsilon''_{ij}$$

$$\chi^{s}_{ij} = \lambda \; \chi^{f}_{ij} + \chi'_{ij} \qquad (3)$$

$$n^{s}_{ij} = \lambda \; n^{f}_{ij} + n'_{ij} + n''_{ij}$$

$$m^{s}_{ij} = \lambda \; m^{f}_{ij} + m'_{ij} \qquad (4)$$

Variables indicated by $(\ )'$ depend linearly on the incremental displacements, while $(\ )''$ indicate quadratic dependence on incremental displacements, as given in Appendix I. Just one load parameter is considered for the whole structure, in the sense that all the loads that are increased until and beyond bifurcation may be characterised by a single parameter. For increasing transverse loading, the external load may be written as

$$p^{s}_{i} = \lambda \; p^{f}_{i} + p'_{i}$$

$$p^{s}_{3} = \lambda \; p^{f}_{3} + p'_{3} \qquad (5)$$

while for increasing axial load,

$$p^{s}_{1} = \lambda \; p^{f}_{1} \qquad (6)$$

The load components p'_{i}, p'_{3} are those which occur in some cases when the structure deforms in the incremental mode. Replacement of (2 - 6) into (1) leads to

$$\pi - \pi_0 = \pi_1 + \pi_2 + \pi_3 + \pi_4 \qquad (7)$$

in which π_0 is the total potential energy in the fundamental state; π_1 contains those terms which are linearly dependent on incremental displacements, and is the first variation of the potential energy; π_2 contains those terms which are quadratic in incremental displacements (associated to the second variation of the functional π); and similarly for π_3 and π_4 . Explicit forms for each of these energy contributions are given in Appendix II.

Since the fundamental state is in equilibrium, the first variation of π must be zero

$$\pi_1 = \delta \pi = 0 \qquad (8)$$

Similarly, if an incremental state of equilibrium exists, then the secondary path may be evaluated from the non-linear system

$$\delta\pi_2 + \delta\pi_3 + \delta\pi_4 = 0 \qquad (9)$$

To obtain the bifurcation point, one may restrict the incremental displacements to be infinitesimal, in which case higher order terms in incremental displacements can be neglected and the load parameter λ at bifurcation is obtained from

$$\delta\pi_2 = 0 \qquad (10)$$

The field of incremental displacements which is associated to the lowest value of λ determined from the eigen-system (1) is the bifurcation mode. This, however, provides no information on the nature of the secondary path.

Following Koiter's theory, the initial curvature of the post-buckling path results from the non-linear equation

$$\delta\pi_2 + \delta\pi_3 = 0 \qquad (11)$$

The FSM will be applied to represent displacement fields and thus obtain approximate solutions to equations (9 - 11).

BOUNDARY CONDITIONS AND FINITE STRIP DISPLACEMENT FIELDS

In the FSM, the displacement field is specified by overall shape functions in one direction (usually trigonometric functions) as in the Ritz method, and by local polynomial functions in the other direction, as in the finite element method. The most complicated part in the choice of displacement functions is the analytical function, which must satisfy:

i) Compatibility along the junctions between plates;

ii) End boundary conditions.

For the analysis of folded plate structures, Cheung [2] has proposed an element called LO2 (lower order, 2 nodal lines) in which linear functions are used for membrane displacements, while the u_3 displacement is interpolated by cubic polynomials. Most of instability studies using the FSM are based on this element, and simple supported boundary conditions simulating diaphragms are satisfied at both ends. Higher order elements could also be used, but the LO2 element has proved to be efficient for stability problems. The trigonometric functions used depend on the equations that have to be approximated. Thus, for the linear fundamental path and for critical loads, the LO2 element as described by Cheung [2] is convenient, but in the post-buckling path, different functions are needed. If the perturbation technique is applied, then each set of equations requires different displacement fields according to the order of the perturbation set.

In the following, the specific functions used in the literature wil be discussed.

Compatible Field [2-5, 12]

In its original version, the LO2 element defined the following interpolation functions:

$$u_1 = \sum_{m=1}^{M} \phi_1^m \cos \frac{m \pi x1}{l}$$

$$u_2 = \sum_{m=1}^{M} \phi_2^m \sin \frac{m \pi x1}{l} \qquad (12)$$

$$u_3 = \sum_{m=1}^{M} \phi_3^m \sin \frac{m \pi x1}{l}$$

in which the polynomial functions ϕ_k^m are given by

$$\phi_1^m = N^1 u_1^{1m} + N^2 u_1^{2m}$$

$$\phi_2^m = N^1 u_2^{1m} + N^2 u_2^{2m}$$

$$\phi_3^m = N^3 u_3^{1m} + N^4 u_3^{2m} + N^5 \beta_2^{1m} + N^6 \beta_2^{2m}$$

with

$$N^1 = 1 - \eta \quad , \quad N^2 = \eta$$

$$N^3 = 1 - 3\eta^2 + 2\eta^3 \quad , \quad N^4 = 3\eta^2 - 2\eta^3 \qquad (14)$$

$$N^5 = \frac{1}{a}(\eta - 2\eta^2 + \eta^3) \quad , \quad N^6 = \frac{1}{a}(\eta^3 - \eta^2)$$

and

$$\eta = \frac{x_2}{a} \qquad (15)$$

At $x = 0$ and $x = l$, the boundary conditions

$$u_3 = u_2 = \frac{\partial^2 u_3}{\partial x_1^2} = 0 \qquad (16)$$

are satisfied; and at the junction between plates compatibility of displacements is preserved since u_2 and u_3 are defined by the same trigonometric functions.

The element degrees of freedom (d.o.f.), in a local coordinate system, are

$$u^m = \{ u_1^{1m}, u_2^{1m}, u_3^{1m}, \beta_2^{1m}, u_1^{2m}, u_2^{2m}, u_3^{2m}, \beta_2^{2m} \} \qquad (17)$$

Use of appropriate transformation matrices R leads to a set of d.o.f in the global system:

$$U^m = R \; u^m \tag{18}$$

in which

$$U^m = \{ U_1^{1m},\; U_2^{1m},\; U_3^{1m},\; \beta_2^{1m},\; U_1^{2m},\; U_2^{2m},\; U_3^{2m},\; \beta_2^{2m} \} \tag{19}$$

The displacement field defined by eqn. (12 - 19) has been used to evaluate bifurcation loads in References [3 - 5].

In the evaluation of the secondary path, Sridharan [13, 15, 17] has applied a perturbation technique to the differential equations which govern the problem, and obtained the following interpolation for the second order displacement field:

$$\begin{aligned}
u_1 &= \sum_{m=1}^{M} \phi_1^m \sin 2m \frac{\pi x_1}{l} \\
u_2 &= \sum_{m=1}^{M} \phi_2^m \cos 2m \frac{\pi x_1}{l} + u_2^o \\
u_3 &= \sum_{m=1}^{M} \phi_3^m \cos 2m \frac{\pi x_1}{l} + u_3^o
\end{aligned} \tag{20}$$

Notice that although eqn.(20) may satisfy compatibility along the junctions between plates, it is uncapable of satisfying the simply supported boundary conditions at the ends. For local buckling this is not a severe limitation [13, 15] , and it has been used for interactive buckling with success [17] .

Incompatible Fields [6, 8, 13]

A different set of trigonometric functions has been used in [6, 13] for the post buckling analysis of plate assemblies under axial loading, and in which buckling occurs in local modes:

$$\begin{aligned}
u_1 &= \sum_{m=1}^{M} \phi_1^m \sin m \frac{\pi x_1}{l} + \lambda \left(\frac{1}{2} - x_1 \right) \\
u_2 &= \sum_{m=1}^{M} \phi_2^m \cos m \frac{\pi x_1}{l} \\
u_3 &= \sum_{m=1}^{M} \phi_3^m \sin n \frac{\pi x_1}{l}
\end{aligned} \tag{21}$$

If the perturbation technique is used to evaluate the post buckling path, the displacement functions (21) are suitable to satisfy the second order in-plane equilibrium equations, in which the effect of the u_3 displacements cannot be neglected [6] . However, compatibility at the junctions between plates cannot be mantained because u_2 and u_3 are represented by different trigonometric series. For local buckling, the end boundary conditions may represent the actual ends of the structure, or the extent of the local buckle. But if no significant global displacements occur, the u_3 displacement at the junctions may be neglected, with the consequence that in-plane and out-of-plane displacements are now uncoupled between plates and the conditions

$$u_3 = N_{22} = 0 \tag{22}$$

are satisfied at junction lines. This, in turn, introduces some restrictions to the type of structures that may be analyzed. No global coordinate system is defined for this element, and all variables are treated in local systems.

Another incompatible field has been introduced by Hancock [8], again for local buckling under axial loads, in the form

$$\begin{aligned} u_1 &= \phi_1 \sin\frac{\pi x_1}{l} \cos\frac{\pi x_1}{l} \\ u_2 &= \phi_2 \left(\sin\frac{\pi x_1}{l}\right)^2 \\ u_3 &= \phi_3 \sin\frac{\pi x_1}{l} \end{aligned} \tag{23}$$

in which l is the extent of the local buckle.

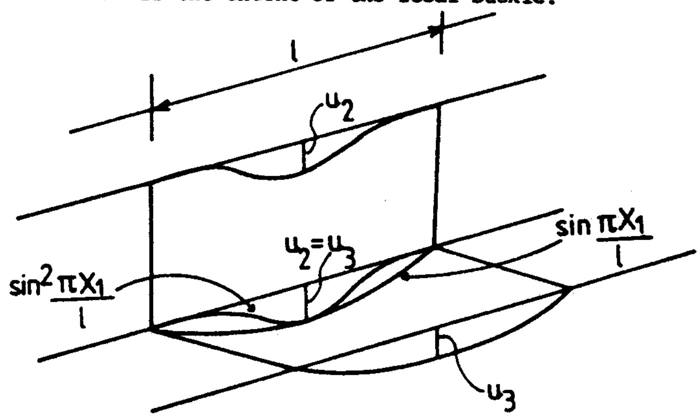


Figure 2

Figure 2 shows that compatibility of displacements (23) at the junction between two orthogonal plates can only be obtained at certain points. For angles other than $\pi/2$, incompatibility at the junction becomes more severe. According to Hancock [8] for right angle junctions the lack of compatibility does not influence the post buckling path for loads lower than twice the bifurcation load.

The introduction of incompatible displacement fields is associated to the reduction in the number of d.o.f. that may be obtained.

BIFURCATION LOADS

Eigenvalue Problem

Bifurcation buckling loads for plate assemblies may be obtained from the second variation of potential energy, eqn.(10), which is often written in the general form

$$(K - \lambda K_G) \phi = 0 \tag{24}$$

where K is the linear stiffness matrix of the plate assembly, K_G is the load-geometry matrix; and ϕ is the eigenvector associated to the eigenvalue λ. Matrices K and K_G are calculated for each element as

$$K = \int_{x_1} \int_{x_2} B^T \ D \ B \ dx_1 \ dx_2 \tag{25}$$

$$K_G = \int_{x_1} \int_{x_2} G^T \ \sigma \ G \ dx_1 \ dx_2 \tag{26}$$

in which B is the linear strain-displacement matrix; D is the elasticity matrix; G the geometry matrix, containing the non linear terms in incremental displacements; and σ the matrix of stresses in the fundamental state. For simple supported plate assemblies, the integrations required in (13 - 14) may be carried out in an explicit form.

For buckling under purely axial load, the eigenvalue problem is expressed as is eqn.(24). For buckling under increasing axial load, but with non-zero lateral loads, eqn.(24) should be written as

$$(K + K_{LL} - \lambda K_G) \phi = 0 \tag{27}$$

in which K_{LL} is a matrix due to the presence of lateral load.

Under purely lateral load, flexural buckling may be modelled by eqn.(24). For buckling under increasing lateral load but with non-zero axial loads, the following eigenvalue problem results:

$$(K + K_{AL} - \lambda K_G) \phi = 0 \tag{28}$$

where K_{AL} is a matrix due to the presence of axial load.

Lateral-torsional buckling under lateral loads produce the eigenvalue problem:

$$[K - \lambda (K_G + K_p)] \phi = 0 \tag{29}$$

in which K_p is a matrix associated to the second order loads that exist in the incremental state of displacement of the structure [3]. For lateral-torsional buckling under transverse water loading, a new effect has to be considered due to the movement of the liquid as the cross section rotates. This produces a new matrix K_w, and eqn.(10) may now be stated as [3]

$$[(K + K_w) - \lambda K_G] \phi = 0 \tag{30}$$

If self weight is considered together with increasing water loading, eqn.(30) is written as

$$[(K + K_w + K_p) - \lambda K_G] \phi = 0 \tag{31}$$

Calculation of matrices K_{LL}, K_{AL}, K_p, K_w can be found in the literature [3 - 5] , and simple explicit forms may be obtained in most cases to improve the efficienty of the computations.

The computational effort to obtain the solution by the FSM is basically associated to the number of coupled harmonics that have to be included in the analysis. As such, we may distinguish between problems in which the deflected shape of the buckled structure may be represented by one harmonic component, and by a number of coupled harmonics.

Single harmonic analysis [3, 4, 12, 18, 20, 21]

In most problems in which global (bifurcation) buckling modes occur, the eigenmodes may be approximated by the first harmonic component of the displacement field. This is the case of instability in a flexural or a lateral-torsional mode under transverse uniform loading, and of global buckling modes under axial loads. Such a solution may also represent a good approximation for global buckling under partial lateral load; and for local buckling problems in which the length of the plate assembly is larger than the extent of local buckle.
As an example of the use of the FSM in evaluation of critical loads, Fig. 3a shows an angle section beam, which is supported on diaphragms at the ends. The beam is made of reinforced concrete, with $t = 0.06$ m, $l = 25$ m, each plate being 1.38 m wide. The complete cross section is discretized using 8 strips and one harmonic component, $m = 1$. The load is applied as a self weight and is constant in the longitudinal direction; thus, eqn.(29) is solved. Buckling occurs in a torsional mode with insignificant changes in the shape of the cross section. Fig. 3b shows results of section critical moments at mid-span, Mc, as a function of angle α of the plates, and it is there seen that the critical moment increases with α until $\alpha = 60°$, and for $\alpha > 60°$ the critical moment decreases; thus, the largest torsional stiffness in this example is obtained for $\alpha = 60°$. As a reference solution, the analytical results for constant moment loading from Meck [11] are also indicated in Fig. 3b; they are seen to agree reasonably well with the FSM in view of the differences in the fundamental state of stresses assumed in each solution.

As an example of the use of eqn.(31), the angle section beam under increasing water loading studied in [3] is reproduced in Fig. 4. Solution of the eigenvalue problem considering a fundamental state of stresses produced by unit dead weight yields a value of $\lambda = 7.27$, and a critical dead weight $P_c = 7.27$ kN/m is obtained. The buckling mode corresponds to a rotation of the cross section with very small

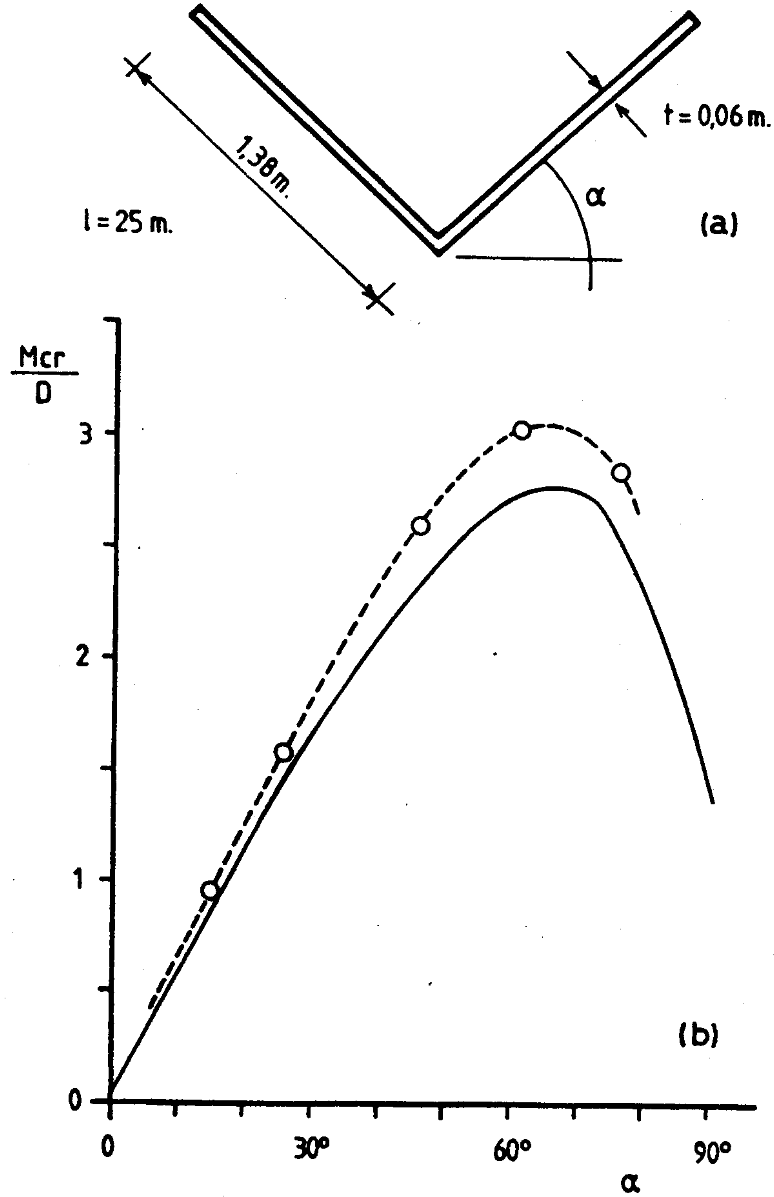
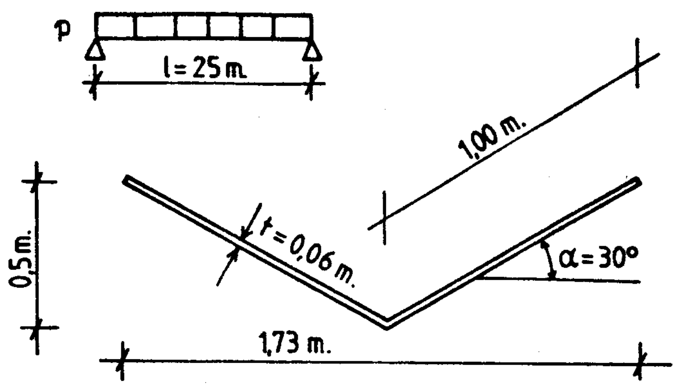


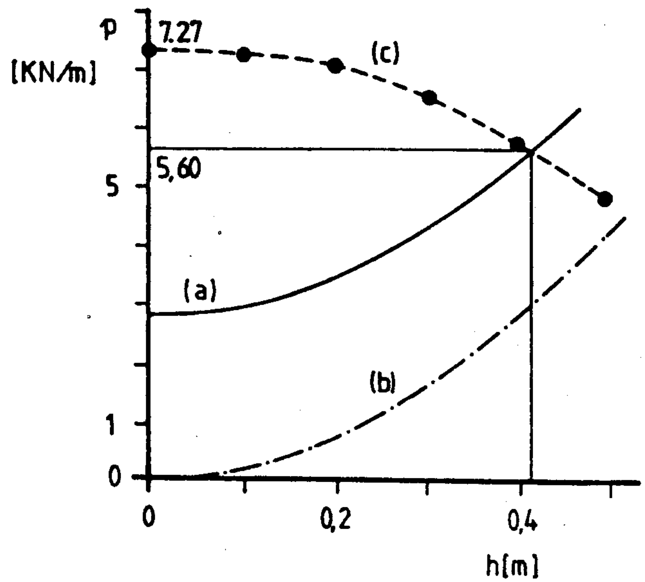
Figure 3 - Torsional buckling of angle section beam under dead weight.

(a) Data for the problem;

(b) Critical sectional moment.
---Finite Strip Method; ——Analytical, Meck [11].



(a)



(b)

Figure 4 - Lateral-torsional buckling under water loading.

(a) Data for the problem;

(b) Critical load

—— eqn. (32); —·—·— eqn. (32) for $p_d=0$;

--- critical load. From Ref. [3].

out of plane deformations, and the angle of rotation follows a half sine wave between supports.

Let h be the water level at a certain loading condition; the total load per unit length applied to the beam is given by

$$p = p_d + \gamma_w h^2 \tan \alpha \qquad (32)$$

where p_d is the self weight of the beam; and γ_w the specific weight of the liquid. For buckling under water loading both, eqn.(31) and (32) have to be satisfied, that is, the eigenvalue λ is associated to a certain water height h. Both equations could be solved simultaneously as an eigenvalue problem subject to restrictions; but it is simpler to evaluate separate solutions and thus obtain the load state that is common to both. This is illustrated in Fig. 4b, and it is seen that both curves intersect at a value $h = 0.42$ m, for which buckling is predicted at $P_c = 5.6$ kN/m. The influence of water loading in this particular case is to reduce the buckling load due to dead weight by 23%.

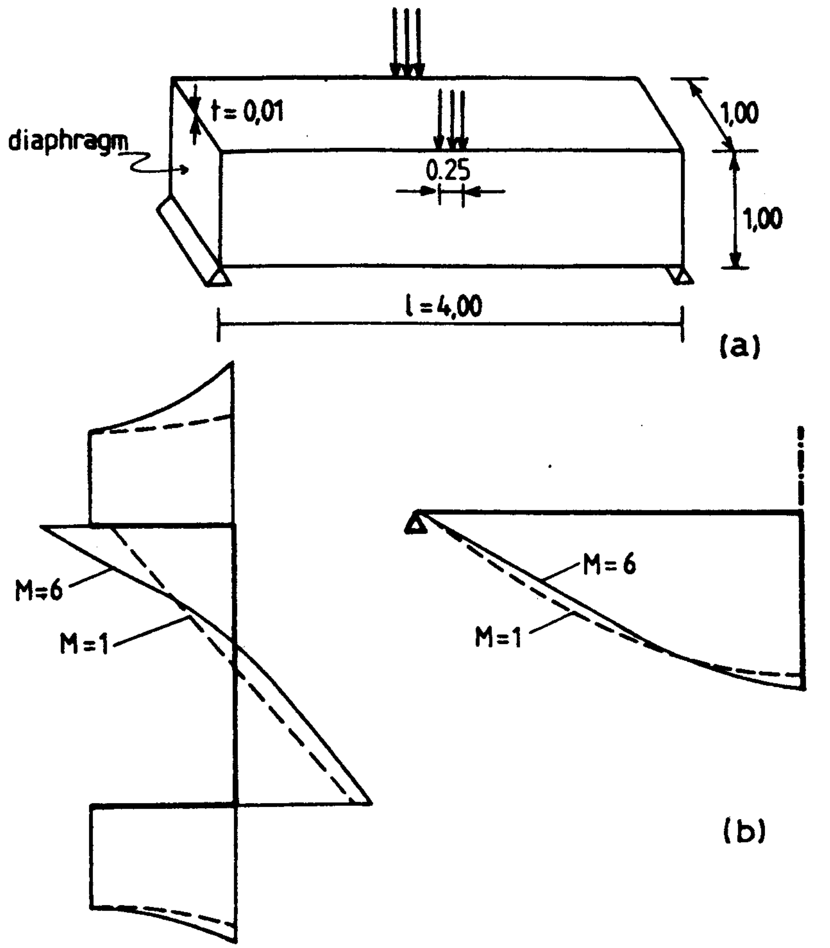
The reduction in buckling load when movements of water are taken into account depends on a number of factors, such as the geometry of the cross section, the length between supports and the self weight of the beam. For the angle section beam studied, if dead weight of the beam is neglected, the curve of applied load is reduced and buckling cannot occur.

Other examples of finite strip applications to lateral-torsional buckling of thin walled open section beams may be seen in [3 - 4]. Of particular interest to the designer is the parametric study of different section profiles, ranging from an angle V section to a U section beam. It has been shown [4] that buckling loads for U beams (both, dead weight and water loaded) are higher than for V beams, but maximum reduction in buckling loads due to water influence is produced in U section beams.

Coupled harmonic analysis [5]

For certain loading conditions, thin-walled plate assemblies must be studied with the aid of more than one harmonic component. In the linear fundamental path, the use of m harmonic components results in m uncoupled systems of equations. However, for evaluation of critical loads, if the fundamental state of stresses is defined as a linear combination of m harmonics, then there is a single eigenvalue problem in which the m harmonics are coupled.

Fig. 5 shows a box girder under partial web loading, studied by Graves Smith and Sridharan [5] using the FSM. The fundamental state of stress resultants n_{11}^f has been obtained in this case for m = 1 and m = 6 harmonics (Fig. 5b), and it may be seen that the differences are significant (of the order of 50%) only at certain points: in particular, the m=6 solution models better the stress field at the upper corners of the box girder, where the loading is applied. The results for critical load in Fig. 5c show that good approximations may be obtained with only 8 strips to model half of the structure; and that use of 3 harmonics produces critical loads with errors of the order of 4%, which is lower than the maximum local errors in the fundamental state.



$$P_{cr} = k \frac{\pi^2 D}{d}$$

$d = 1$

M	k	error
3	3.327	3.9 %
4	3.227	0.8 %
5	3.205	0.2 %
6	3.200	--

(c)

Figure 5 - Box girder under partial web loading.

(a) Data for the problem;

(b) Stress Resultant n_{11}^{f} at mid-span and at lower conrners;

(c) Critical loads for different number of coupled harmonics. From Ref. [5].

For this particular problem, even a single harmonic analysis would provide a good estimate of the bifurcation load, and this is true for most problems in which the load is symmetric with respect to mid-span. But if the load is non-symmetric, then larger differences should be expected between single and coupled harmonic analysis, and it is in those problems that coupling should not be neglected.

Bifurcation under axial loads [5, 12, 18, 20, 21]

The FSM had its first applications in buckling analysis in column-type plate assemblies loaded by edge axial compression. The work by Wittrick and coworkers [12, 18, 20] was based on a single harmonic analysis, and covered both isotropic and orthotropic structures. Yoshida [21] applied the FSM to compute bifurcation loads in stiffened plates, and also developed special beam strips to model eccentric stiffeners. The formulation by Graves Smith and Sridharan [5] can also take into account buckling under axial loading.

POST BUCKLING ANALYSIS

Single Mode Post Buckling Path [6, 13 - 15]

Once the bifurcation load parameter λ_c has been obtained, the secondary path can be evaluated, the problem being one of finding equilibrium states along a path. Unlike the fundamental path, which is usually considered as linear, the secondary path is always non linear and its determination is more complicated than the primary equilibrium state. The FSM has also been used to compute such non linear problem defined by eqn.(9).

The first application of the FSM for post buckling analysis is due to Graves Smith and Sridharan [6]. The perturbation technique was applied to the non linear equations, and the variables expressed as

$$\begin{aligned} \lambda &= \lambda^{(0)} + \lambda^{(1)}\xi + \lambda^{(2)}\xi^2 + \dots \\ u_i &= u_i^{(0)} + u_i^{(1)}\xi + u_i^{(2)}\xi^2 + \dots \end{aligned} \qquad (33)$$

in which

$$(\ )^{(r)} = \frac{d^r (\ )}{d\,\xi^r} \cdot \frac{1}{r!} \qquad r = 1, \dots p \qquad (34)$$

and ξ is the perturbation parameter used to describe the secondary path. The parameter ξ is chosen as the amplitude of the buckling mode. Replacement of (33) into (9) leads to a set of linear equations which may be solved in a sequential order. The displacement fields $u_i^{(r)}$ were approximated by incompatible finite strips defined by eqn.(21). Examples of plate assemblies which exhibit stable post buckling behaviour were presented for axially loaded structures, using one and two harmonic solutions.

A compatible element for post buckling analysis was developed on the same lines in Ref.[13]. For this second element, compatibility of displacements along the junctions is satisfied, so that it is possible to study the influence of junctions displacements on the secondary path. "Crinkly" collapse of the corners in box columns could thus be detected with the FSM.

Non uniform compressive loads were studied by Sridharan [14] using the incompatible displacement field. Two modes of loading were considered: prescribed load eccentricity and prescribed end displacements, and the results showed that the behaviour of plate structures is significantly affected by the mode of loading.

In the previous studies, only local buckling modes were investigated, in the sense that the junctions between plates do not have significant displacements and remain almost straight. Application of the compatible strip to secondary paths involving displacement of the junctions was presented by Sridharan [15], with special reference to local-torsional buckling of open cross section columns.

Mode Interaction [2, 16, 17]

In the previous section, the computation of a post buckling path for single buckling modes using the FSM was reviewed. In some cases, however, single mode analyses do not lead to reasonable results because of coupling between modes. Interactions between modes are of special interest in thin-walled metal structures whenever some feature of the behaviour is modified and would not be predictable in terms of separate analysis of each mode. Of particular interest in metal structures are [16]

(i) Interaction between local modes;

(ii) Interaction between a local and a local-torsional mode; and

(iii) Interaction between a local and global mode (either flexural or flexural-torsional global mode).

The theory of mode interaction stems from the work of Koiter [10], and the displacements are written as

$$u = u^{(0)} + u^{(1)}\xi_1 + u^{(2)}\xi_2 + u^{(11)}\xi_1^2 + u^{(22)}\xi_2^2 + u^{(12)}\xi_1\xi_2 + \dots \quad (35)$$

in which ξ_1, ξ_2 are taken as the amplitudes of the two modes considered; $u^{(\ )}$ are the displacement fields associated to the fundamental path $u^{(0)}$; to the first order eigenmode solution $u^{(1)}$; to the second order solution $u^{(11)}$; and so on. Separate solutions have to be obtained for each displacement field $u^{(\ )}$ before the values of ξ, ξ_2 can be solved in eqn.(35).

Sridharan has used the FSM to obtain discrete solutions to eqn.(35) in problems of doubly symmetric interactions under uniform end compression. Mode interaction of local and global buckling in stiffened panels [16]; of local and lateral-torsional buckling in T-section beams under end moments [2]; and of I-section columns [17] using the FSM have been reported in the literature. The extension to mode interaction under suddenly applied load is also considered in [17].

CONCLUSIONS

The basic features of finite strip formulations applied to stability problems have been presented. A literature review shows that the technique has been successfully applied to evaluate bifurcation loads under transverse and under axial loading; to determine a secondary path; to study mode interaction in the post critical range; to compute a non-linear fundamental path with initial imperfections [8], and to consider plasticity effects [9].

Some assessment of the economies that may be achieved with the FSM in instability analysis are given by Yoshida in terms of computing time [21] and by Graves Smith and Sridharan in terms of d.o.f. and computing effort [5], with the result that in some cases the FSM requires 100 times less computer time than the finite element method.

APPENDIX I

Kinematic and Constitutive Equations

The linear kinematic relations, in a local coordinate system, may be written as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{36}$$

$$X_{ij} = -\frac{\partial \beta_i}{\partial x_j} = -\frac{\partial^2 u_3}{\partial x_i \partial x_j} \tag{37}$$

Equations (36), (37) are used to evaluate a linear fundamental state, and for linear components of deformation in incremental displacements. The non-linear components of deformation, E''_{ij} are expressed as

$$E''_{ij} = \frac{1}{2}\,\frac{\partial u_r}{\partial x_i}\,\frac{\partial u_r}{\partial x_j} \qquad (r = 1,\ 3) \tag{38}$$

However, not all terms in eqn.(38) are of the same magnitude. If the plate assembly is long, non-linear terms involving the u_1 component are negligible, and in many applications also terms containing the u_2 component may be neglected. In local buckling modes, only terms which are non-linear in u_3 are relevant. In global modes both u_3 and u_2 should be considered because the joints between plates have significant displacements. Thus, the following simplified non-linear components of deformation may be adopted in the FSM for general purposes:

$$E''_{11} = \frac{1}{2}\left[\left(\frac{\partial u_3}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2\right]$$

$$E''_{22} = \frac{1}{2} \left[\left(\frac{\partial u_3}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right] \tag{39}$$

$$E''_{12} = \frac{1}{2} \left[\frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_2}{\partial x_2} \right]$$

Sridharan [14, 15] has shown that the most significant term involving u_2 is $\frac{1}{2} \left(\frac{\partial u_2}{\partial x_1}\right)^2$ whenever in-plane displacements of the plates occur.

For an isotropic and homogeneous material, the constitutive equations of linear elasticity result in

$$n_{ij} = \frac{Et}{1+\nu} \left(E_{ij} + \frac{\nu}{1-\nu} \delta_{ij} E_{ll}\right) \tag{40}$$

$$m_{ij} = \frac{E t^3}{12(1+\nu)} \left(X_{ij} + \frac{\nu}{1-\nu} \delta_{ij} X_{ll}\right) \tag{41}$$

$$(i, j, l = 1,2)$$

in which E is the elastic modulus and ν Poisson's ratio.

APPENDIX II

Incremental terms in total potential energy

The terms on the right hand side in eqn.(7) are written as:

$$\pi_1 = \sum_{k=1}^{K} t \lambda \{ \int_{x_1} \int_{x_2} [\frac{1}{2} (n^f_{ij} E'_{ij} + m^f_{ij} X'_{ij} + n'_{ij} E^f_{ij} + m'_{ij} X^f_{ij})$$

$$- (p^f_i u_i + p^f_3 u_3)] \, dx_1 \, dx_2$$

$$- \int_{x_2} [P^f_1 u_1]_{x_1=0}^{x_1=l} \, dx_2 \} \tag{42}$$

$$\pi_2 = \sum_{k=1}^{K} t \{ \int_{x_1} \int_{x_2} [\frac{1}{2} (n'_{ij} E'_{ij} + m'_{ij} X'_{ij}) + \frac{\lambda}{2} (n^f_{ij} E''_{ij}$$

$$+ n''_{ij} E^f_{ij}) - (P'_i u_i + P'_3 u_3)] \, dx_1 \, dx_2 \} \tag{43}$$

$$\pi_3 = \sum_{k=1}^{K} t \{ \frac{1}{2} \int_{x_1} \int_{x_2} (n''_{ij} E'_{ij} + n'_{ij} E''_{ij}) \, dx_1 \, dx_2 \} \tag{44}$$

$$\pi_4 = \sum_{k=1}^{K} t \{ \frac{1}{2} \int_{x_1} \int_{x_2} (n''_{ij} E''_{ij}) \, dx_1 \, dx_2 \} \tag{45}$$

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