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REINJECTION PROBABILITY FUNCTION WITH LOWER BOUND OF THE REINJECTION FOR INTERMITTENCY TYPE III

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Abstract. Intermittency is characterized by the successive occurrence of a signal that alternates chaotic burst between quasi-regular periods or laminar phases. The intermittency phenomenon is a continuous route from regular to chaotic motions and it is classified into three types in function of the local Poincaré map and the value of the respective Floquet multiplier (local property): type I, type II and type III. To determine the intermittency behavior and characteristic parameters such as the average laminar length it is necessary to know the reinjection probability function or RPF (global property). At the present several books and papers consider that the RPF is constant or some artificial function. However there are some tests in which theses conditions are not satisfy. In this paper is introduced a new methodology to obtain the reinjection probability function for intermittency type III with and without lower bound of the reinjection. This technique permits to reach a more general RPF and it includes the constant RPF as a particular case. The proposed technique to obtain the RPF shows advantages because it reduces the noise in experimental and numerical data. Two maps are analyzed to prove the accuracy of the new RPF function and the viability of the proposed technique to obtain it.

1. INTRODUCTION

Intermittency is a particular form of deterministic chaos, in which transition between laminar and chaotic phases occurs. A system is in regular behavior until, with a small change in a parameter, it begins to show chaotic burst at irregular intervals. Pomeau and Maneville introduce the intermittency concept relates to the Lorenz system (Maneville and Pomeau, 1979 and Pomeau and Maneville, 1980). There are several topics into applied mechanics in which intermittency is present: Lorenz system, periodically forced non linear oscillators, hydrodynamics, Rayleigh-Bénard convection; derivative non-lineal Schoendinger equation, turbulence, etc.

Intermittency phenomenon is classified into three types: I, II and III according to the Floquet multipliers or eigenvalue in the local Poincaré map. For continuous-time system, the intermittency type I arises in a cyclic-fold bifurcation in which a stable and an unstable orbits collapse, then the system lost the stable orbits in the vicinity of the vanished periodic orbits. For maps, intermittency type I occurs by means an inverse tangent bifurcation, in this case an eigenvalue leaves the unit circle through +1. Intermittency type II begins in a subcritical Hopf bifurcation, so that two complex-conjugate Floquet multipliers or two complex-conjugate eigenvalues of the local Poincaré map exit the unit circle. Intermittency type III is related with a subcritical period-doubling or flip bifurcation and one Floquet multiplier leaves the unit circle through -1. When one-dimensional map, $f(x)$, presents intermittency type III in any point $x = x_f$, the Schwartzian derivative must be positive in this point (Nayfeh and Balachnadran, 1995):

$$S\ f(x) = \frac{\dfrac{\partial^3 f(x)}{\partial x^3}}{\dfrac{\partial f(x)}{\partial x}} - 1.5 \left[\frac{\dfrac{\partial^2 f(x)}{\partial x^2}}{\dfrac{\partial f(x)}{\partial x}} \right]^2 > 0 \tag{1}$$

In this paper we present a new reinjection probability function for intermittency type III together with the methodology to calculate it. The local Poincaré map for intermittency type III is:

$$x_{n+1} = -(1+\varepsilon)\ x_n - a\ x_n^3 \quad \text{with} \quad a > 0 \tag{2}$$

The intermittency exists only for $\varepsilon > 0$ (Kin et al., 1997).

We study two non-linear maps for intermittency type III. The first map is given by the following equation:

$$x_{n+1} = F(x) = -(1+\varepsilon)\ x_n - a\ x_n^3 + b\ x^c\ \sin(x) \tag{3}$$

The origin $x = 0$ is always a fixed point. It is stable for $-2 < \varepsilon < 0$ and it is unstable for $\varepsilon > 0$. We study this map using $c = 6$, in this case the map presents intermittency type III. For $x = 0$, in Eq.(3) $\partial^3 f(x)/\partial x^3$ and $\partial f(x)/\partial x$ are negative and $\partial^2 f(x)/\partial x^2 = 0$, then the Schwartzian derivative is positive and there is a inflexion point.

Figure 1 shows the map (3) for three values of b = 0.5, 2 and 5. The reinjection mechanism is dependent on the points, x_r, in which $dF(x_r)/dx = 0$. The map has two x_r points; both are symmetric with respect the origin. Points x_0 initially close to the fixed point, $x = 0$, iterate away of the origin in a process drives by parameters ε, a and the cubic exponent. When x_n is near to the x_r points the reinjection mechanism began and it needs two iterations for reinjection.

The second map was presented by Laugesen et al. (1997):

$$x_{n+1} = G(x) = \left[-(1+\varepsilon)x_n - x_n^3 \right] e^{-bx^2} \tag{4}$$

The map posses one fixed point at $x = 0$. The slope at the fixed point is (-1-ε), then the point $x = 0$ is stable for $-2 < \varepsilon < 0$. For $\varepsilon = 0$ and $b < 1$ exists a subcritical period doubling bifurcation and for $\varepsilon > 0$ there is intermittency. When x becomes large, the factor e^{-bx^2} becomes small producing an efficient reinjection mechanism. The map given by Eq. (4) in showed in Fig. 2 for different values of parameter b = 0.1, 0.175, 0.25.

For both maps, Eqs.(3) and (4) the Schwartzian derivative is negative for the point $x = 0$, because the third and first derivatives posses the same sign (they are negative), and the second derivative is null for $x = 0$.

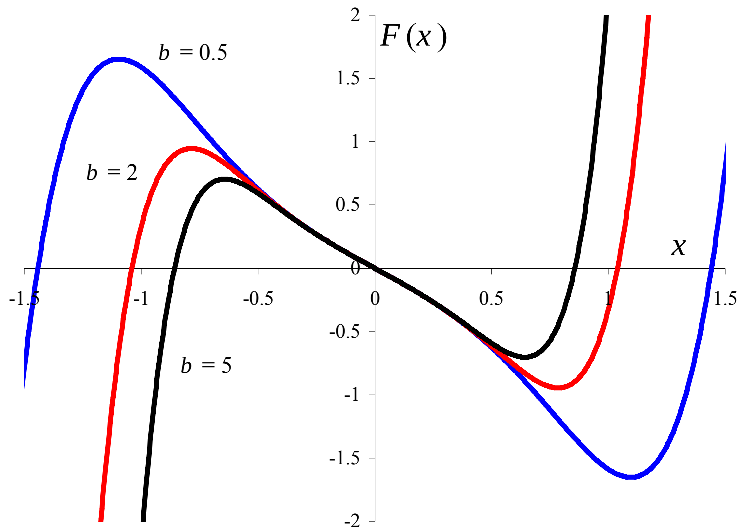

Figure 1. Function $F(x)$ of the map given by Eq.(3) for b = 0.5, 2 and 5.

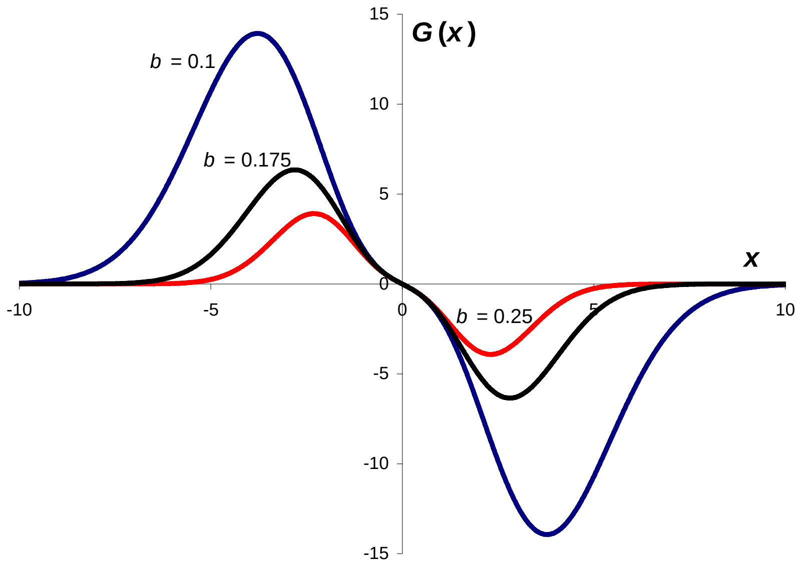

Figure 2. Function $G(x)$ of the map given by Eq.(4) for $b = 0.1$, 0.175 and 0.25.

In Fig. 3 we show, for Eq.(3), the bifurcation diagram considering b variable and $\varepsilon = 0.01$, $a = 1$ and $c = 6$. We observe that for $b > 1.07$ the reinjection near of the fixed point is not produced, there is a hole, without reinjection, around the fixed point.

Figure 4 shows the bifurcation diagram for Eq.(4) with $a = 1$ and $\varepsilon = 0.1$. For $b > 0.14$ there are not reinjection around the fixed point, and a hole appears.

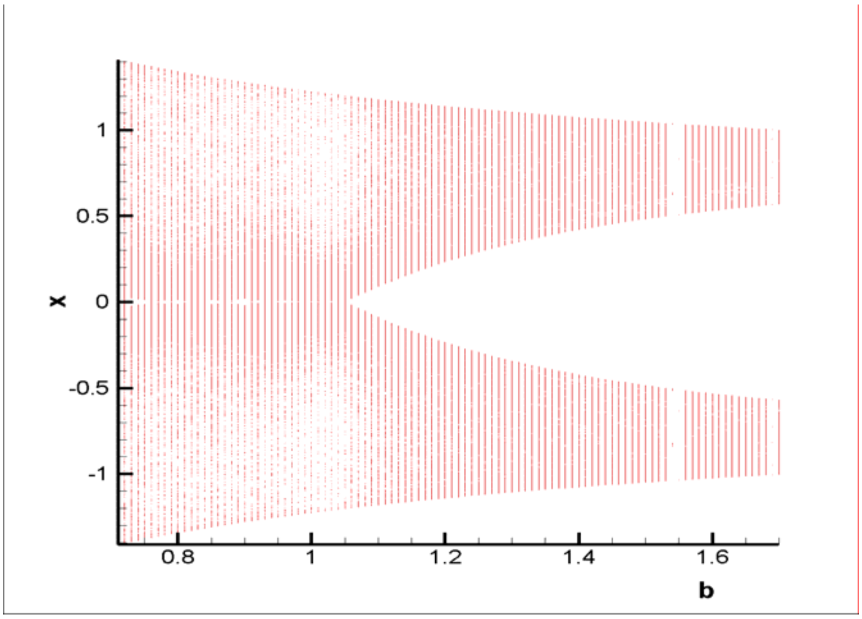

Figure 3. Bifurcation diagram for Eq.(3). $\varepsilon = 0.01$, $a = 1$ and $c = 6$

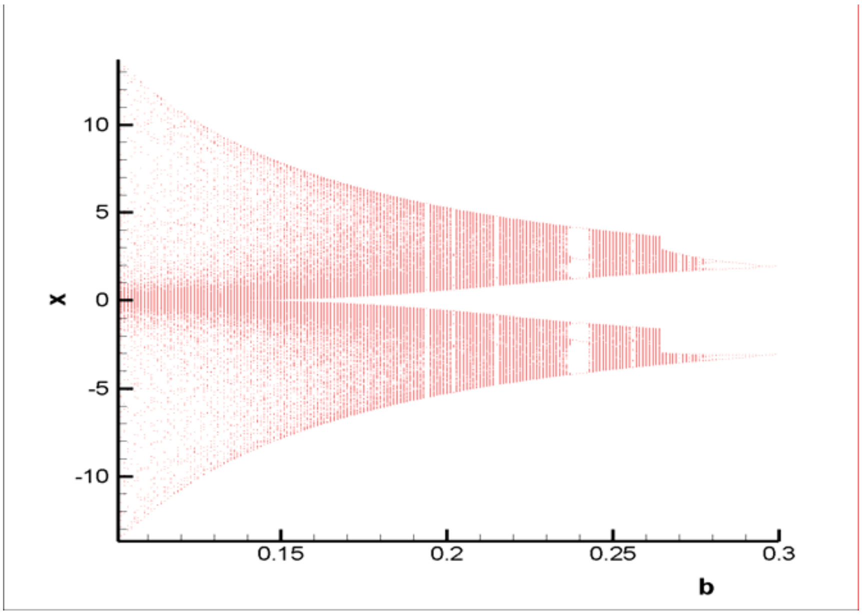

Figure 4. Bifurcation diagram for Eq.(4). ε = 0.1 and *a* = 1.

Figures 5 and 6 show the temporal variation and the relative frequency of laminar lengths for Eqs.(3) and (4) respectively. Fron both pictures it is possible to note the typical dynamics evolution for systems with intermittency type III.

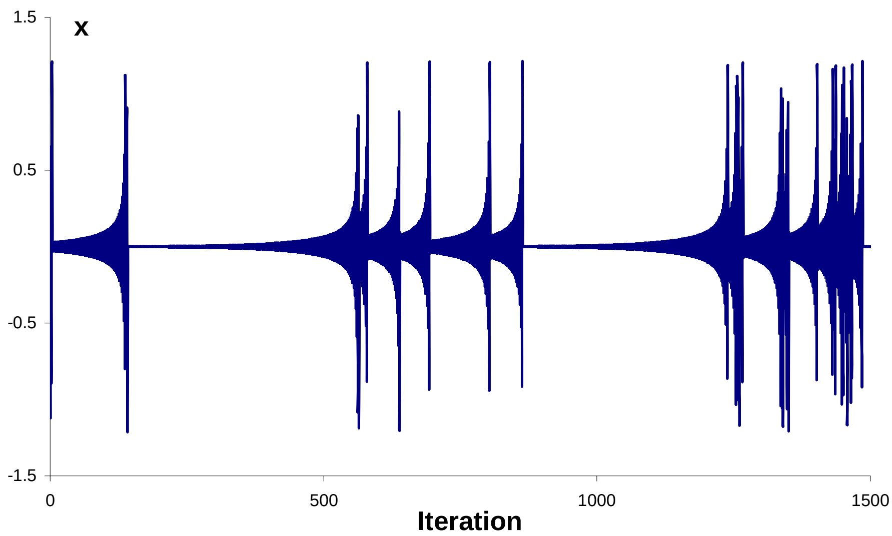

Figure 5. Temporal variation for Eq.(3). ε = 0.01, b = 1.05 and *a* = 1.

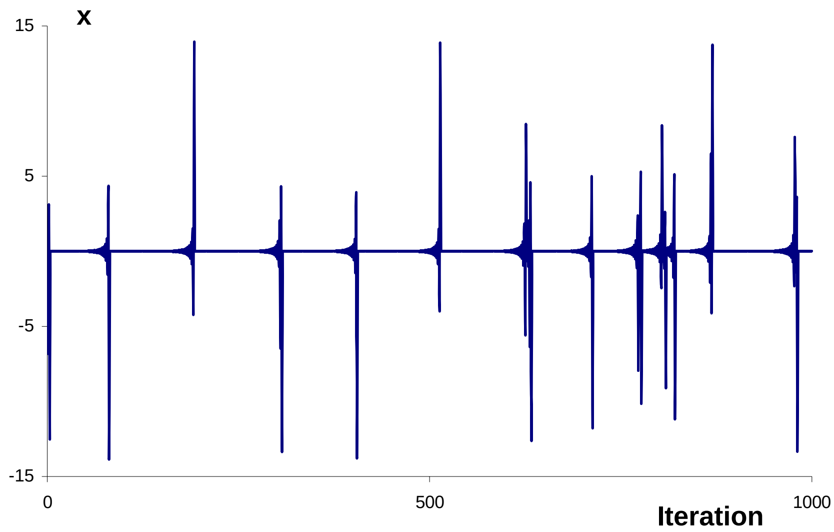

Figure 6. Temporal variation for Eq.(4). ε = 0.1, b = 0.1 and *a* = 1.

2. REINJECTION PROBABILITY DISTRIBUTION

It is clear that the reinjection probability density $\phi(x)$ of the system from chaotic burst into the laminar zone is governed by the chaotic behavior of the system, and then it depends on each particular system or map. The local Poincaré map of the intermittency does not give the necessary information to determine the reinjection probability density (RPD). In general it is very difficult to obtain analytically $\phi(x)$. Also it is difficult to get $\phi(x)$ experimentally or numerically, because the large number of data needed to cover each interval of length δx in the reinjection region and the noise introduced during the numerical evaluation or the experimental measurements. Because of this, different approximations have been used in literature to study the intermittent systems. The most commune and simple approximation is to consider $\phi(x)$ uniform and thus independent of the reinjection point (Shuster and Just, 2005). Other approximations have been used for different investigations, for instant, in the paper of Won and Kim (2000) it is assumed that the reinjection is in a fixed point Δ, that is $\phi(x) = \delta(x - \Delta)$, to investigate the effect of noise in type-I intermittency. In other case it was consider $\phi(x) \propto 1/\sqrt{x-\Delta}$ to study type-III intermittency in a electronic circuit (Won, 2003).

In this paper it is not measure the reinjection probability density, $\phi(x)$, directly from the numerical data. Instead of this it is implemented the function $M(x)$ defined as following

$$M(x) = \frac{\int_{x_i}^{x} \tau\, \phi(\tau)\, d\tau}{\int_{x_i}^{x} \phi(\tau)\, d\tau} \quad if \quad \phi(\tau) \neq 0; \qquad M(x) = 0 \quad if \quad \phi(\tau) = 0 \tag{5}$$

where x_i is the closed point to the fixed point with reinjection, *i.e.* the lower bound of the reinjection. The range of the upper integral limit is $[x_i\,,\,c]$, which defines an specific and acceptable laminar region.

The function $M(x)$ was numerically calculated in a broad class of maps obtaining, in good approximation, the lineal form

$$M(x) = m\,x + x_h \tag{6}$$

The last equation is a generalization of the function introduced by del Río and Elaskar (2009). From Eq.(5) it possible to determine that $M(x_i) = x_i$, and it is verifies the following linear relation for the function $M(x)$

$$M(x) = m\left(x - x_i\right) + x_i \tag{7}$$

In a previous work del Río and Elaskar (2009), showed that the slope m plays an important role in the intermittency dynamics and the function $M(x)$ is an accurate tool to study systems with intermittency type II. Note that for uniform probability reinjection, $m = 0.5$ and $x_i = 0$, so that $M(x) = 0.5\,x$. Eq.(7) does not impose restrictions about the slope m, and it can be bigger or less than the 1/2. However, in all numerical developed test were find $0 < m < 1$.

Figures 3 and 4 show there are values of b satisfying $x_i \neq 0$ for equations (3) and (4). In these cases the maps will not have reinjection near of the unstable fixed point and the function $M(x)$ will be given by Eq.(7).

According with previous results, it is assumed that $M(x) = m\,x + x_i$, then the reinjection probability density can be written as:

$$\phi(x) = \lambda\left(x - x_i\right)^{\alpha} \quad \mathit{with} \quad \alpha = \frac{1-2m}{m-1} \tag{8}$$

where λ is determined by the normalization condition, for intermittency type III is:

$$\int_0^c \lambda\, x^{\alpha} dx = 0.5 \tag{9}$$

because the function $M(x)$ evaluates only half of the complete laminar region. Assuming $\alpha > -1$, or equivalent $0 < m < 1$, the integrals converge and we get for λ the expression:

$$\lambda = \frac{1}{2}\frac{\alpha+1}{\left(c - x_i\right)^{\alpha+1}} = \frac{1}{2}\,\frac{m}{1-m}\left(c - x_i\right)^{m/(m-1)} \tag{10}$$

Eq.(7) shows that, in the linear approximation of $M(x)$, the density $\phi(x)$ is determined only by the parameters m and x_i. Both parameters are easier to measure that the complete function $\phi(x)$. Note that the shape of $\phi(x)$ can be very different from the flat line (uniform reinjection).

Figure 7 shows the function $M(x)$, for map given by Eq.(3), for two cases. The parameters utilized in both examples are: $a = 1$, $c = 6$, $\varepsilon = 0.01$, $b = 1.05$ and 1.1. It is possible to observe, in both examples, that the curves $M(x)$ are approximately right lines and the slope m is clearly defined in each case. However exist an important difference between the two examples, the test with b = 1.05 satisfies $x_i = 0$ and the test with b = 1.1 verifies $x_i \neq 0$. The slopes m are evaluated using minimal square technique, for b = 1.05 is obtained m = 0.36241 and for b = 1.1 m and x_i are 0.3686 and 0.0525 respectively. Then, the functions $M(x)$ can be written as:

$$M(x) \cong 0.36241\, x \qquad for \qquad b = 1.05 \tag{11a}$$

$$M(x) \cong 0.3686\, x + 0.0525 \qquad for \qquad b = 1.1 \tag{11b}$$

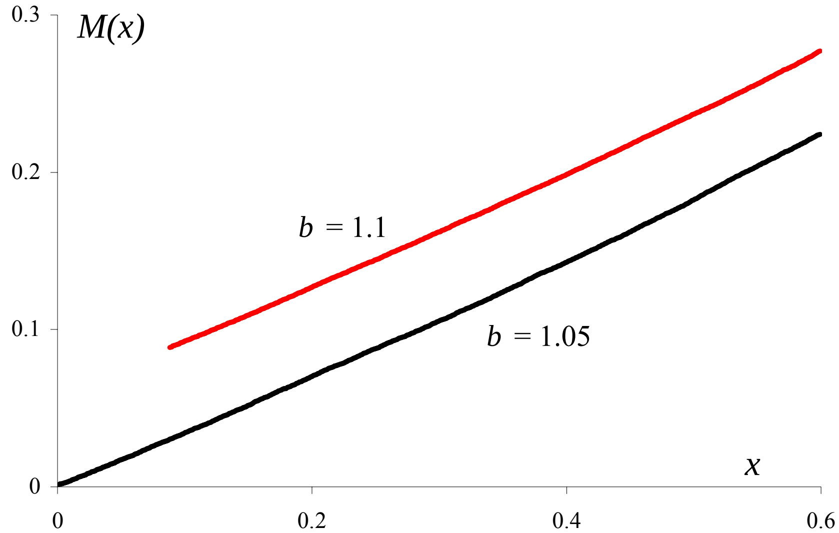

Figure 7. Function $M(x)$ for the map given by Eq.(3), with b = 1.05 and 1.1, a = 1, c = 6, ε = 0.01.

We can evaluate the reinjection probability function using the Eqs.(7, 10, 11a, 11b). The RPF for corresponding to Eq.(11a) can be written as:

$$\phi(x) = 0.38\, x^{-0.43} \tag{12a}$$

The RPF for Eq.(11b) is:

$$\phi(x) = 0.431\left(x - 0.088\right)^{-0.416} \tag{12b}$$

Figures 8 and 9 show, for Eq.(3), the numerical reinjection probability in good agreement with $\phi(x)$ evaluated from Eqs.(12a) and (12b). The parameters are the same that we use in Fig. 5. Dots were obtained by numerical evaluation; the continuous line were obtained by the plot of the Eqs.(12a) and (12b).

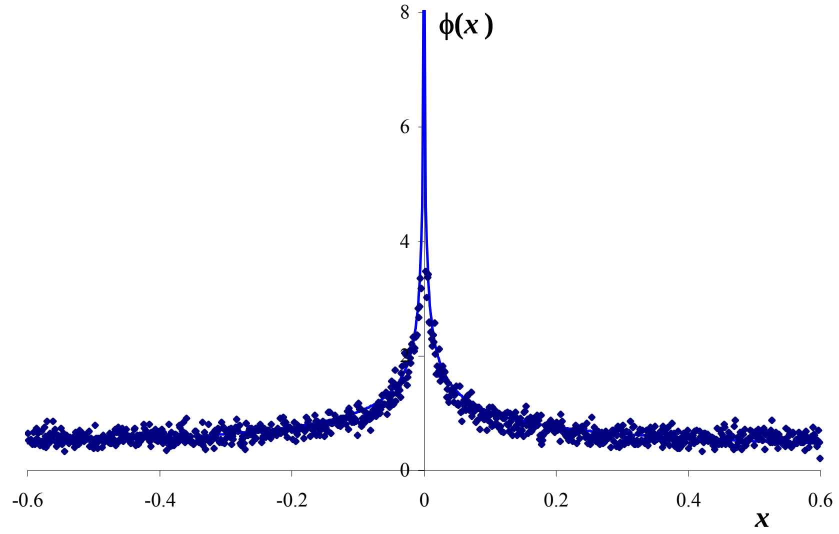

Figure 8. Reinjection probability density for Eq.(3). The parameters are b = 1.05, a = 1, c = 6, ε = 0.01. Dots indicate numerical evaluations and continuous line show the Eq.(12a)

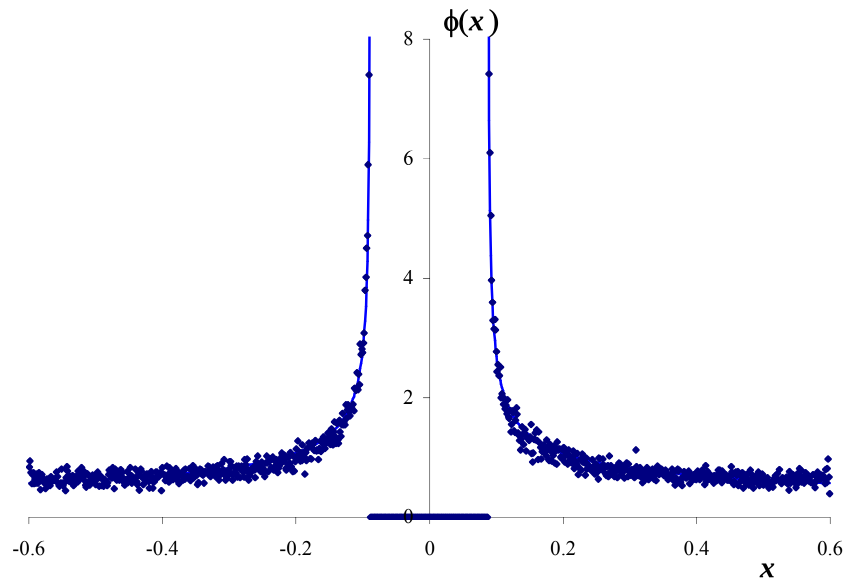

Figure 9. Reinjection probability density for Eq.(3). The parameters are b = 1.1, a = 1, c = 6, ε = 0.01. Dots indicate numerical evaluations and continuous line show the Eq.(12b)

Figure 10 shows the function $M(x)$, for Eq.(4), for two cases. The parameters are: a = 1, ε = 0.1, b = 0.1 and 0.18. $M(x)$ are approximately right lines and the slope m is clearly defined in each case. In similar form from the map (3), for this map exists cases that satisfies $x_i = 0$ and $x_i \neq 0$. For b = 0.1 the reinjection is produced around the fixed pint, however for b = 0.18 it does not occur. The slopes m are evaluated using minimal square, for b = 0.1 is obtained m = 0.187 and for b = 0.18 m and x_i are 0.26 and 0.22 respectively. Then, the functions $M(x)$ are:

$$M(x) \cong 0.187\, x \qquad \mathit{for} \qquad b = 0.1 \tag{13a}$$

$$M(x) \cong 0.26\, x + 0.22 \qquad \mathit{for} \qquad b = 0.18 \tag{13b}$$

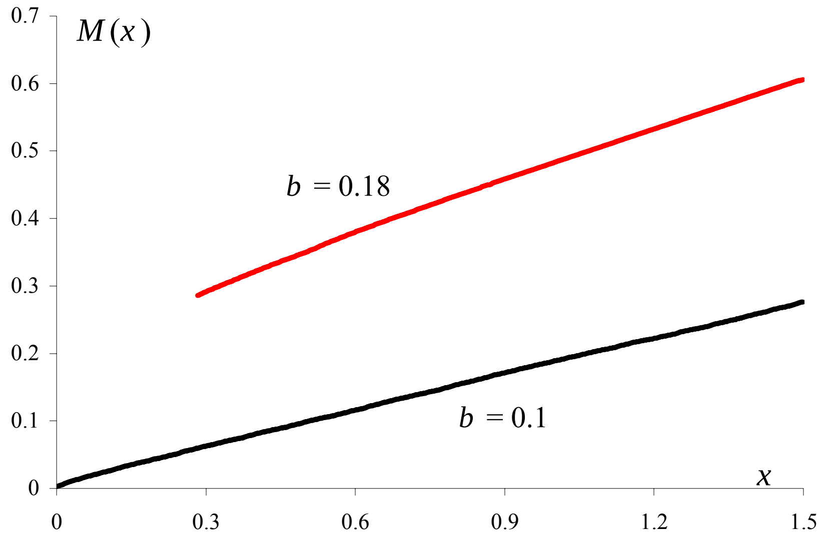

Figure 10. Function $M(x)$ for Eq.(4), with $b = 0.1$ and 0.18, $a = 1$, $\varepsilon = 0.01$.

We can evaluate the reinjection probability function using the Eqs.(7, 10, 13a, 13b). The RPF for corresponding to Eq.(13a) can be written as:

$$\phi(x) = 0.105\, x^{-0.77} \tag{14a}$$

The RPF for Eq.(13b) is:

$$\phi(x) = 0.164\left(x - 0.2835\right)^{-0.649} \tag{14b}$$

Figures 11 and 12 show, for Eq.(4), the numerical reinjection probability in good agreement with $\phi(x)$ evaluated from Eqs.(14a) and (14b). The parameters are the same that we use in Fig. 10. Dots were obtained by numerical evaluation; the continuous line were obtained by the plot of the Eqs.(14a) and (14b).

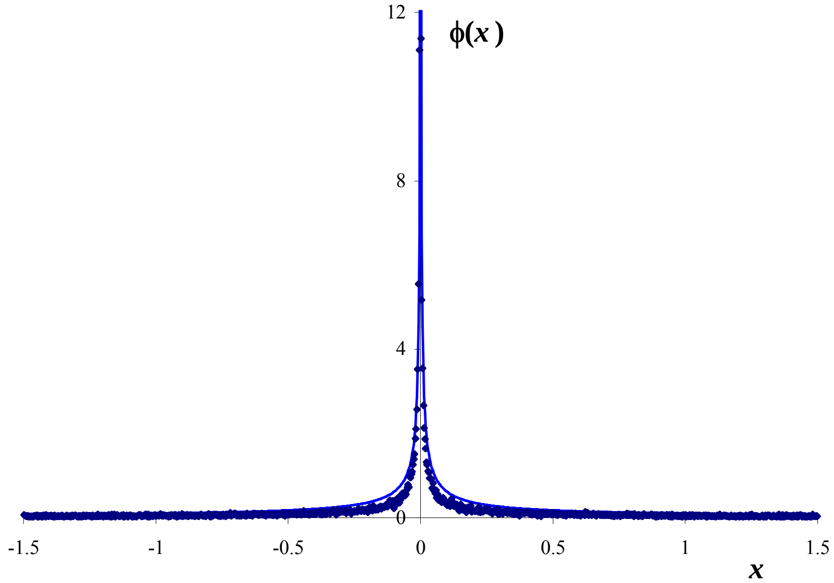

Figure 11. Reinjection probability density for Eq.(). The parameters are b = 0.1, a = 1, ε = 0.1. Dots indicate numerical evaluations and continuous line show the Eq.(11a)

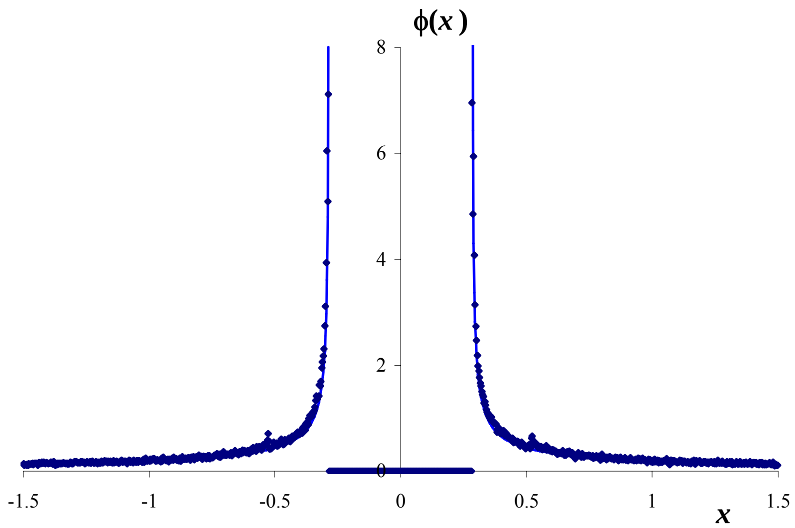

Figure 12 Reinjection probability density for Eq.(2). The parameters are b = 0.18, a = 1, ε = 0.1. Dots indicate numerical evaluations and continuous line show the Eq.(11b)

It is important to note that the continuous function, $M(x)$, posses the capacity to filter the usual noise of the numerical data, hence this function gets better description of the reinjection probability density function.

4. CONCLUSIONS

In this paper is presented the function $M(x)$ as a useful tool to determine the reinjection probability density function, $\phi(x)$, in intermittency type II and III.

The function $M(x)$ is easier to obtain, numerical and experimentally, than the reinjection probability density because the noise is reduced.

In several cases $M(x)$ is linear, $M(x) = m(x - x_i) + x_i$, then it is possible to find that the reinjection probability density has an exponential form, $\phi(x) = \lambda\,(x\text{-}x_i)^{\alpha}$. This last expression is a generalization of the usual uniform reinjection probability density approximation, which correspond to $\alpha = 0$ or $m = 1/2$ and $x_i = 0$.

The concordance between the numerical results with the function $\phi(x) = \lambda\,(x\text{-}x_i)^{\alpha}$ is very good. Inclusive in the cases in which exists a lower bound of the reinjection during the reinjection process the function $M(x)$ has shown to be an accurate tool to determine the reinjection probability density function.

If $x_i = 0$ the functions $M(x)$ and $\phi(x)$ are reduced to $M(x) = mx$ and $\phi(x) = \lambda\, x^{\alpha}$ respectively as such as were presented by del Río and Elaskar (2009).

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