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BEAM EMBEDDED IN A WINKLER SOIL WITH CLEARANCES. ANALYTICAL SOLUTION OF THE NONLINEAL BENDING PROBLEM.

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Abstract. The nonlinear bending problem of a beam embedded in a Winkler soil with clearances is addressed by means of an analytical approach. This problem is of interest, for instance, in the study of the drillstrings behavior under certain load conditions. Usually, a drillstring is modeled as a bar inside an outer cylinder (bore-hole wall) with clearances that add strong nonlinearities. This work is part of a wider study on drillstrings and a paper on the nonlinear vibration of this type of structure was presented in ENIEF 2006. Here the title problem is simplified to a plane bar making contact with a Winkler-type soil. The governing differential problem is derived using a minimal energy principle. The unknowns are the lateral displacement due to bending (as usual) and the length of contact. The consideration of the latter unknown leads to special restrictions among the admissible directions within the Calculus of Variation deriving in a particular statement of the problem. Once the differential problem is fully established, a solution of pairs of load-length of contact is found. Two particular examples are worked out: a cantilever beam with a lateral tip load and a simply supported beam subjected to external end bending moments, in both cases with several soil stiffness values. The numerical results are compared with a finite element model. The availability of an analytical approach permits the calibration of other numerical solutions, e.g. the study of convergences issues. On the other hand other complexities, such as the consideration of axial loads including self-weight that leads to the inclusion of the second order effect, are under study at present.

1 INTRODUCTION

The title problem is of interest when studying the beam-columns (typically drillstrings) structural behavior. One of the possible states deals with loads leading to lateral displacements of a beam inside a hole surrounded by soil. The present work is part of a wider study on drillstrings and a paper on the nonlinear vibration of this type of structure was presented in ENIEF 2006 Filipich et al. (2006). The Winkler soil model has been object of wide research. An interesting review (Dutta and Roy, 2002) deals with the simpler and more complex models employed in the study of soil-structures problems. A recent paper by ElGanainy and ElNaggar (2009) deals with a nonlinear Winkler foundation model and Silveira et al. (2008) tackle the equilibrium and stability of structural elements under unilateral constraints using a Ritz type approach. A simplified approach may be tackled by means of a Winkler elastic soil model. The authors have used this model to address some beam dynamic problems (Filipich and Rosales, 1988, 2002; Filipich et al., 2006). Despite the simplicity of the problem, certain peculiarities arise, in the present study, in the derivation of the governing equations within the Calculus of Variation regarding with the variable limits of the contact. Also, the clearances are responsible of a nonlinear response. Some parts of the beams will be in contact with the soil while others remain inside the hole. The length of the contact region is, at first, unknown for a given load state. The governing equations and the boundary conditions are derived and special care should be taken to consider the existence of integration limits that are functions of the independent variable. Even in the range of small deformations and under linear elastic behavior of the material, unilateral constraints lead to highly nonlinear responses. The problem will be stated for two different configurations, a cantilever beam with a transverse tip load and a simply supported beam with bending external moments at the ends. Finally numerical examples will be presented to illustrate the problem and comparison with finite element results also included.

2 STATEMENT OF THE PROBLEM: BEAM IN A BORE-HOLE SURROUNDED BY A WINKLER-TYPE SOIL.

In this Section, the governing differential system of a beam inserted in a hole which is surrounded by a Winkler-type soil is stated. Two different configurations will be studied, i.e. a uniform cantilever beam with a transverse load P and a simply supported beam subjected to external end bending moments.

2.1 Uniform cantilever beam with a transverse load P.

Let the total static energy U^* of the beam with a lateral displacement due to a lateral tip load and considering the existence of clearance ε (see Figure 1), be

$$\begin{aligned} 2U^*[v_1;v_2;a] &= EJ\left\{\int_0^a v_{1xx}^2 dX + \frac{k_0}{EJ}\int_0^a (v_1-\varepsilon)^2 dX + \right. \\ &\left. \int_a^L v_{2xx}^2 dX - \frac{2P}{EJ} v_1(0)\right\} \\ &(0 \le X \le L) \end{aligned} \tag{1}$$

where $v_j = v_j(X)$, $j = 1, 2$, $a = a(X)$, E is the Young's modulus, J is the moment of inertia, P is the lateral load and k_0 is the Winkler soil constant. The equilibrium condition is

$$\delta U^* = 0 \tag{2}$$

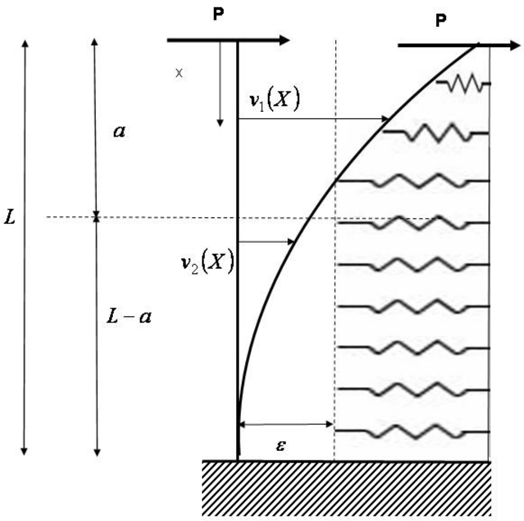

Figure 1: Cantilever beam in hole of radius ε surrounded by a Winkler soil of stiffness k_0 subjected to a tip lateral load P.

If "$f_j(X)$" and "$b(X)$" denote the admissible directions of v_j and a respectively, and we write

$$2U^*[v_1+\eta f_1; v_2+\eta f_2; a+\eta b] = EJ\left\{\int_0^{a+\eta b}(v_{1xx}+\eta f_{1XX})^2 dX + \right.$$
$$\left.\frac{k_0}{EJ}\int_0^{a+\eta b}(v_1+\eta f_1-\varepsilon)^2 dX + \int_{a+\eta b}^{L}(v_{2xx}+\eta f_{2XX})^2 dX - \frac{2P}{EJ}[v_1(0)+\eta f_1(0)]\right\} \quad (3)$$

Then, from the Calculus of Variation and condition (2),

$$\delta U^* = \frac{d}{d\eta}\left\{U^*[v_1+\eta f_1; v_2+\eta f_2; a+\eta b]\right\}\Big|_{\eta=0} = 0 \quad (4)$$

which leads to the following condition

$$\int_0^a v_{1xx}f_{1xx}dX + \frac{k_0}{EJ}\int_0^a (v_1-\varepsilon)f_1 dX + \int_a^L v_{2xx}f_{2xx}dX -$$
$$\frac{P}{EJ}f_1(0) + b\left\{v_{1xx}^2(a) + \frac{k_0}{EJ}[v_1(a)-\varepsilon] - v_{2xx}^2(a)\right\} = 0 \quad (5)$$

The terms into curly brackets originates on the derivation of the integral with limits variable with η.

Note 1 Recall (Gelfand and Fomin, 1963; Fox, 1987; Russak, 1996) that if

$$I = I(\eta) = \int_{p(\eta)}^{q(\eta)} \varphi(\eta, u)du$$

then the derivative of the integral w.r.t. η is

$$I_\eta = \frac{dI(\eta)}{d\eta} = \int_{p(\eta)}^{q(\eta)} \varphi_\eta(\eta, u)du + \varphi(\eta, q(x))q_x(x) - \varphi(\eta, p(x))p_x(x).$$

Before working with Eq. (5), let us impose the conditions for the geometrical continuity (compatibility of deformation conditions) at a

$$v_1(a) = \varepsilon \tag{6a}$$
$$v_2(a) = \varepsilon \tag{6b}$$
$$v_{1X}(a) = v_{2X}(a) \tag{6c}$$

Now, before the integration by parts of Eq. (5), it will be necessary to know, among other variations, $f_j(a)$ and $f_{jX}(a)$, $j = 1, 2$ (variations of the function and the first derivative, respectively) which deserves an special detail. In effect, if we write, for instance, the following equalities:

$$v_1(a) \equiv F_1[v_1; a] \equiv \frac{1}{a}\int_0^a v_1(a)dX \tag{7a}$$
$$v_2(a) \equiv F_2[v_2; a] \equiv \frac{1}{L-a}\int_a^L v_2(a)dX \tag{7b}$$

The definitions of the "functionals" $F_1[v_j, a]$ permits to express the following variations:

$$\delta v_j(a) = \left.\frac{d}{d\eta}\{F_j[v_j + \eta f_j; a + \eta b]\}\right|_{\eta=0} \quad (j = 1, 2) \tag{8}$$

where, for instance, the variation of F_1 writes

$$F_1[v_1 + \eta f_1; a + \eta b] = \frac{1}{a + \eta b}\int_0^{a+\eta b} [v_1(a + \eta b) + \eta f_1(a + \eta b)]dX.$$

The following relevant conclusions are derived:

$$\delta v_j(a) = f_j(a) + v_{jX}(a)b \tag{9}$$
$$\delta v_{jX}(a) = f_{jX}(a) + v_{jXX}(a)b \tag{10}$$

NOTE 2 It could be thought at first that according to conditions (6a) and (6b) and with $v_1(a)$ and $v_2(a)$ already imposed, then one should conclude that $f_1(a) = f_2(a) = 0$, which is erroneous, as is now demonstrated. If the latter were used, wrong boundary conditions would be obtained. From Eqs. (6)

$$\delta v_j(a) = 0 \quad (j = 1, 2) \tag{11a}$$
$$\delta v_{1X}(a) = \delta v_{2X}(a) \tag{11b}$$

with which, after using results (9) and (10), one obtains

$$f_j(a) = -v_{jX}(a)b \quad (j = 1, 2) \tag{12a}$$
$$f_{2X}(a) = f_{1X}(a) + b[v_{1XX}(a) - v_{2XX}(a)]. \tag{12b}$$

Now, integrating by parts Eq. (5) and taking into account (6a), the following expression is obtained

$$\int_0^a \left[v_{1XXXX} + \frac{k_0}{EJ}(v_1 - \varepsilon) \right] \; f_1 dX + \int_a^L v_{2XXXX} \, f_2 dX +$$
$$|v_{1XX} \;\; f_{1X}|_0^a - |v_{1XXX} \;\; f_1|_0^a - P \; f_1(0) + |v_{2XX} \;\; f_{2X}|_a^L -$$
$$|v_{2XXX} \;\; f_2|_a^L + b[v_{1XX}^2(a) - v_{2XX}^2(a)] = 0 \tag{13}$$

On the other hand, when dealing with a cantilever, two stable or geometric boundary conditions have to be fulfilled

$$v_2(L) = 0 \tag{14a}$$
$$v_{2X}(L) = 0 \tag{14b}$$

that, in this case, give place to

$$f_2(L) = f_{2X}(L) = 0 \tag{15}$$

After accepting that the variations $f_j(X)$ and their derivatives are independents from the Fundamental Theorem of the Calculus of Variation, and considering (9), (10) and (15), we arrive to the conclusion that the equilibrium (13) is equivalent to the fulfillment of the following equations

$$v_{1XXXX} + \frac{k_0}{EJ}(v_1 - \varepsilon) = 0 \tag{16a}$$
$$v_{2XXXX} = 0 \tag{16b}$$
$$v_{1XX}(0) f_1(0) = 0 \tag{16c}$$
$$\left[v_{1XXX}(0) - \frac{P}{EJ} \right] f_1(0) = 0 \tag{16d}$$
$$[v_{1XX}(a) - v_{2XX}(a)] \, f_{1X}(0) = 0 \tag{16e}$$
$$\{[v_{1XXX}(a) - v_{2XXX}(a)] \, v_{1X}(a) + [v_{1XX}(a) - v_{2XX}(a)] \, v_{1XX}(a)\} \, b = 0 \tag{16f}$$

As may be observed, the DE (16a) and (16b) and the natural boundary conditions at $X = 0$ and the continuity ones at $X = a$, must be fulfilled. Meanwhile, conditions (16c-16e) would appear commonly in a domain arbitrary divided at $X = a$, with $k_0 = 0$. Eq. (16f) is not apparent. Anyway, in the present case of the cantilever beam, in general, $v_1(0) \neq 0$ and $v_{1X}(0) \neq 0$ and consequently, $f_1(0) \neq 0$ and $f_{1X}(0) \neq 0$. Also, in general, $v_{1X}(a) \neq 0$ and then, due to Eqs. (6) and (12b), $f_{1X}(a) \neq 0$ as well as $b \neq 0$ (it has not to be assumed null in general). Additionally with the above- mentioned and, from (16), the following should be satisfied,

$$v_{1XX}(0) = 0 \tag{17a}$$
$$v_{1XXX}(0) = \frac{P}{EJ} \tag{17b}$$
$$v_{1XX}(a) = v_{2XX}(a) \tag{17c}$$
$$v_{1XXX}(a) = v_{2XXX}(a) \tag{17d}$$

Equations (17) together with Eqs. (6) and (14) yield the nine boundary and continuity conditions for this problem.

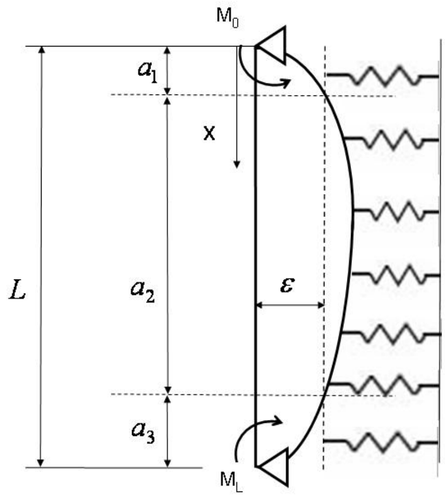

Figure 2: Simply supported beam in hole of radius ε surrounded by a Winkler soil of stiffness k_0 subjected to end moments M_0 and M_L.

NOTE 3 As a complement to NOTE 2, let us say that, should the problem had been considered without the variation of the limit $X = a$, the correct conditions (17d) would not have been obtained. On the other hand, this equation is simply the continuity of the shear force for both regions. However, due to requirements (6a) and (6b) the conditions had to be deduced by a more complex and conceptual way.

The unknowns are also nine: the eight constants of integration of the ODE's (16a) and (16b) and the limit a between the regions with and without contact with the soil. The resulting problem is strongly nonlinear. There is more than one solution, but only one is of interest in the engineering problem.

2.2 Simply supported beam subjected to external bending moments.

The approach is completely similar to the above-stated. Figure 2 depicts the geometry and loads of the present problem. Now three different regions are dealt with, i.e. a_1, a_2 and $a_3 = L - a_1 - a_2$. The total potential energy writes

$$2U^*[v_1; v_2; v_3; a_1; a_2; a_3] = EJ\left\{\int_0^{a_1} v_{1xx}^2 dX + \int_{a_1}^{a_1+a_2} v_{2xx}^2 dX + \right.$$
$$\left. \frac{k_0}{EJ}\int_{a_1}^{a_1+a_2} (v_2-\varepsilon)^2 dX + \int_{a_1+a_2}^{L} v_{3xx}^2 dX - \frac{2M_0}{EJ} v_{1X}(0) + \frac{2M_L}{EJ} v_{3X}(L) + \right\} \quad (18)$$

After finding the extreme value δU^* the following governing problem arises

$$v_{1XXXX} = 0 \tag{19a}$$
$$v_{2XXXX} + \frac{k_0}{EJ}(v_2 - \varepsilon) = 0 \tag{19b}$$
$$v_{3XXXX} = 0 \tag{19c}$$
$$v_1(0) = 0;\ v_3(L) = 0; \tag{19d}$$
$$v_1(a_1) = \varepsilon;\ v_2(a_1) = \varepsilon;\ v_2(a_1 + a_2) = \varepsilon;\ v_3(a_1 + a_2) = \varepsilon; \tag{19e}$$
$$v_{1X}(a_1) = v_{2X}(a_1);\ v_{2X}(a_1 + a_2) = v_{3X}(a_1 + a_2); \tag{19f}$$
$$v_{1XX}(0) = \frac{M_0}{EJ};\ v_{3XX}(L) = 0; \tag{19g}$$
$$v_{1XX}(a_1) = v_{2XX}(a_1);\ v_{2XX}(a_1 + a_2) = v_{3XX}(a_1 + a_2); \tag{19h}$$
$$v_{1XXX}(a_1) = v_{2XXX}(a_1);\ v_{2XXX}(a_1 + a_2) = v_{3XXX}(a_1 + a_2); \tag{19i}$$

Then, there are 14 unknowns —12 integration constants and the contact lengths a_1 and a_2 and 14 boundary and continuity conditions (19d-19i), that make this nonlinear problem consistent.

3 CANTILEVER BEAM: SOLUTION OF THE DIFFERENTIAL SYSTEM

Now let us tackle the solution of the differential problem stated in Eqs. (16-17). Let us consider that the contact length a is fixed, i.e. assumed known for each equilibrium solution. Let us introduce the next notation

$$0 \le X \le a;\ \ x_1 = \frac{X}{a} \ \Rightarrow\ 0 \le x_1 \le 1;\ \ a \le X \le L;\ \ x_2 = \frac{X-a}{L-a} \ \Rightarrow\ 0 \le x_2 \le 1;$$

Also if $k^4 = k_0 a^4/(EJ)$ and $p = Pa^3/(EJ)$ the following equations yield

$$\left.\begin{array}{l} v_1'''' + k^4 v_1 = k^4 \varepsilon \\ v_1''(0) = 0;\quad v_1'''(0) = p; \end{array}\right\} \text{with } (\cdot)' = \frac{d(\cdot)}{dx_1} \tag{20}$$

$$\left.\begin{array}{l} v_2'''' = 0 \\ v_2(1) = 0;\quad v_2'(0) = 0; \end{array}\right\} \text{with } (\cdot)' = \frac{d(\cdot)}{dx_2} \tag{21}$$

$$v_1(1) = 0;\ v_2(0) = 0; \tag{22a}$$
$$v_2(0) = \varepsilon; \tag{22b}$$
$$\frac{L-a}{a} v_1'(1) = v_2'(0); \tag{22c}$$
$$\frac{(L-a)^2}{a^2} v_1''(1) = v_2''(0); \tag{22d}$$
$$\frac{(L-a)^{3\,3}}{a}\ v_1'''(1) = v_2'''(0). \tag{22e}$$

The solutions of the two unknowns $v_1(x_1)$ and $v_1(x_2)$ with $\lambda \equiv k/\sqrt{2} = \sqrt[4]{k_0/(4EJ)}$, write

$$\begin{aligned} v_1(x_1) &= \varepsilon + \sinh\lambda x_1 \left(A_1 \sin\lambda x_1 + A_2 \cos\lambda x_1\right) \\ &\quad + \cosh\lambda x_1 \left(A_3 \sin\lambda x_1 + A_4 \cos\lambda x_1\right) \end{aligned} \tag{23}$$

$$v_2(x_2) = B_1 + B_2 x_2 + B_3 x_2^2 + B_4 x_2^3 \tag{24}$$

3.1 Practical algorithm to solve the cantilever case

From (20) (second row, left condition), one finds $A_1 = 0$ and using (22b), $B_1 = \varepsilon$. Given the data L, ε, E, k_0 and J, equations (21) and (22) will be used to find $\{A_2 \ \ A_3 \ \ A_4 \ \ B_2 \ \ B_3 \ \ B_4\}$, but functionally depending on the parameter a (not as function of x). From (20),

$$p = p(a) = v_1'''(0) \Rightarrow P = P(a) = \frac{pEJ}{a^3} \tag{25}$$

If the applied load $P = P_0$ is assumed known, then the root a of

$$P_0 - \frac{p(a)EJ}{a^3} = 0 \tag{26}$$

is no more than the sought solution a for each P_0, if a solution exists.

4 SIMPLY SUPPORTED BEAM: SOLUTION OF THE DIFFERENTIAL SYSTEM

The methodology is completely analogous to the above described. Let us consider the configuration of Figure 2 and that the contact lengths a_1 and a_3 are fixed, i.e. assumed known for each equilibrium solution. The solutions are now divided in three spans, The solutions of the two unknowns $v_1(x_1)$ and $v_1(x_2)$ with $\lambda \equiv k/\sqrt{2} = \sqrt[4]{k_0/(4EJ)}$, write

$$\begin{aligned} v_1(x_1) &= A_1 + A_2 x_1 + A_3 x_1^2 + A_4 x_1^3 && (27) \\ v_2(x_2) &= \varepsilon + \sinh \lambda x_2 \left(B_1 \sin \lambda x_2 + B_2 \cos \lambda x_2\right) \\ &\quad + \cosh \lambda x_2 \left(B_3 \sin \lambda x_2 + B_4 \cos \lambda x_2\right) && (28) \\ v_3(x_3) &= C_1 + C_2 x_3 + C_3 x_3^2 + C_4 x_3^3 && (29) \end{aligned}$$

where $x_1 = X/a_1$, $x_2 = (X - a_1)/a_2$, $x_3 = [X - (a_1 + a_2)]/a_3$, $\lambda = a_2 \sqrt[4]{k_0/(4EJ)}$ and $a_1 + a_2 + a_3 = L$. It is accepted that a_1 and a_3 are fixed and considered as parameters. $v_1(x_1)$, $v_1(x_2)$ and $v_3(x_3)$ are found from solving Eqs. (19a-22c) (after non-dimensionalization). Boundary and continuity conditions (19d-19i) after non-dimensionalization yield (with $r_1 \equiv a_2/a_1$ and $r_1 \equiv a_2/a_1$)

$$v_1(1) = 0; \ \ v_2(0) = \varepsilon; v_3(0) = \varepsilon; \tag{30a}$$
$$v_1(1) = v_2(0) \tag{30b}$$
$$v_2(1) = v_3(0) \tag{30c}$$
$$r_1 v_1'(1) = v_2'(0) \tag{30d}$$
$$r_2 v_2'(1) = v_3'(0) \tag{30e}$$
$$v_1''(0) = -m_0 \equiv -\frac{M_0 a_1^2}{EJ} \tag{30f}$$
$$r_1^2 v_1''(1) = v_2''(0) \tag{30g}$$
$$r_2^2 v_2''(1) = v_3''(0) \tag{30h}$$
$$v_2''(1) = -m_L \equiv -\frac{M_L a_3^2}{EJ} \tag{30i}$$
$$r_1^3 v_1'''(1) = v_2'''(0) \tag{30j}$$
$$r_3^3 v_2'''(1) = v_3'''(0) \tag{30k}$$
$$v_3(1) = 0 \tag{30l}$$

4.1 Practical algorithm to solve the simply supported case.

From conditions (30a), $A_1 = 0$, $B_4 = 0$ and $C_1 = \varepsilon$ are respectively found. Given the data L, ε, E, k_0 and J, equations (30b-30e), (30g-30h) and (30j-30l), $\{A_2\ \ A_3\ \ A_4\ \ B_1\ \ B_2\ \ B_3\ \ C_2\ \ C_3\ \ C_4\}$ yield. They functionally depend on the parameters a_1 and a_3 ($\Rightarrow a_2 = L - (a_1 + a_3)$) (not as function of x). A highly nonlinear system of equations is obtained: two equations (30f) and (30i)with two unknowns $m_0(a_1, a_3)$ and $m_L(a_1, a_3)$, i.e.

$$M_0 = M_0(a_1, a_3) = \frac{EJ}{a_1^2} m_0(a_1, a_3) \tag{31}$$

$$M_L = M_L(a_1, a_3) = \frac{EJ}{a_3^2} m_L(a_1, a_3) \tag{32}$$

As before, if the applied moments M_0 and M_L are assumed known, then the values of a_1 and a_3 (if a solution exists) arise from the following nonlinear system:

$$M_0 - \frac{m_0(a_1, a_3)EJ}{a_1^2} = 0 \tag{33}$$

$$M_L - \frac{m_L(a_1, a_3)EJ}{a_3^2} = 0. \tag{34}$$

5 NUMERICAL EXAMPLES

The two cases above presented, a cantilever beam with a lateral tip load and a simply supported beam with moments at the ends, are numerically solved for various values of the involved parameters. Table 1 depicts the cantilever beam results for different values of the lateral load P, length L, soil stiffness k_0 and clearance ε. The resulting values of a found with exact solution herein presented are depicted and compared with values found with a finite element model solved with FlexPDE (PDE Solutions, 2009). In general, the agreement is very good. It should be said that the finite element solution requires of convergence studies to attain reliable values. Also the displacement at the tip of the beam (f) is compared with the displacement of the same beam and load but assuming $k_0 = 0$ (without soil) ($f_0 = PL^3/3EJ$). Finally, the simply sup-

Lateral load P	Beam length L	Soil stiff. k_0	Hole radius ε	Tip displ. (no soil) f_0	Present (exact) solution f	Present (exact) solution a	FE model (FlexPDE) f	FE model (FlexPDE) a
10^6	3	$10^8/70$	10^{-2}	0.0268	0.0265	1.331	0.0264	1.331
10^5	10	$10^8/70$	$3\ 10^{-2}$	0.0992	0.0548	3.042	0.0547	3.042
10^6	5	$10^7/70$	$3\ 10^{-2}$	0.1240	0.1224	2.812	0.1225	2.813
10^5	10	$10^6/70$	$3\ 10^{-2}$	0.0992	0.0975	5.038	0.0977	5.043
10^5	1	$10^8/70$	$5\ 10^{-2}$	$9.920\ 10^{-5}$	$9.917\ 10^{-5}$	0.810	$9.919\ 10^{-5}$	0.810
10^5	10	$10^7/70$	$2\ 10^{-2}$	0.0992	0.0821	5.606	0.0822	5.609
10^5	10	$10^8/70$	$2\ 10^{-2}$	0.0992	0.0454	3.793	0.0454	3.794
10^5	10	$10^8/70$	$2\ 10^{-2}$	0.0992	0.0716	2.020	0.0716	2.023

Table 1: Cantilever beam in a Winkler soil with clearence ε and lateral load P. f is the tip displacement and a is the region with soil contact. Values found with present methodology and with finite elements discretization (PDE Solutions, 2009).

ported beam was numerically solved. The values found with the present approach are reported in Table 2 for diverse prescribed end bending moments M_0 and M_L. The displacement at the center of the beam ($x = L/2$) without soil $f_0 = L^2(M_0 + M_L)/(16EJ)$ is used as a reference value.

Moment M_0	Moment M_L	Beam length L	Soil stiff. k_0	Hole radius ε	Central displ. (no soil) f_0	Present (exact) solution f	a_1	a_3
$7\ 10^6$	$7\ 10^6$	10	$8\ 10^6$	10^{-1}	0.2604	0.1553	1.8468	1.8468
$7\ 10^6$	0	10	$8\ 10^6$	10^{-1}	0.1302	0.1161	2.4478	3.9896
$7\ 10^6$	$7\ 10^6$	10	10^7	$1\ 10^{-1}$	0.2604	0.1483	1.9509	1.9509
$7\ 10^6$	$7\ 10^6$	10	10^8	$1\ 10^{-1}$	0.2604	0.1100	3.1158	3.1158

Table 2: Simply supported beam in a Winkler soil with clearence ε and end moments M_0 and M_L. f is the central displacement and a_1 and a_3 are the region without soil contact. Values found with present methodology.

6 CONCLUSIONS

The nonlinear bending problem of a beam in a hole of radius ε surrounded by a Winkler soil was addressed by means of an analytical approach. This problem is of interest, for instance, in the study of drillstrings behavior under certain load conditions. The discontinuity in the media adds strong nonlinearities. This work is part of a wider study on drillstrings and a paper on the nonlinear vibration of this type of structure was presented in ENIEF 2006. The governing differential problem was derived using a minimal energy principle. Here, besides the usual unknowns (lateral displacements), the length of contact was also involved. The consideration of the latter unknown led to special restrictions among the admissible directions within the Calculus of Variation deriving in a particular statement of the problem. Two particular examples were numerically solved: a cantilever beam with a lateral tip load and a simply supported beam subjected to external end bending moments, in both cases with several soil stiffness values. The results found with the above stated exact solution, were compared with a finite element model. The availability of an analytical approach permits the calibration of other numerical solutions, e.g. the study of convergences issues. On the other hand other complexities, such as the consideration of axial loads including self-weight, are under study at present.

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