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A REALIZABLE CONSTITUTIVE MODEL FOR FIBER-REINFORCED NEO-HOOKEAN SOLIDS

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Abstract. We provide a *closed-form* analytical expression for the effective stored-energy function of Neo-Hookean solids reinforced by a random distribution of anisotropic cylindrical fibers, subject to general finite-strain loading conditions. The expression is obtained by means of a homogenization constitutive theory recently proposed by the authors (Lopez-Pamies O., Idiart M.I. Fiber-reinforced hyperelastic solids: A realizable homogenization constitutive theory. *J. Eng. Math.*, submitted) to determine the mechanical response of fiber-reinforced hyperelastic solids. The central idea in this theory is to devise a special class of random microgeometries —by means of an iterated homogenization procedure together with an exact dilute result for sequential laminates— that allows to compute exactly the macroscopic response of the resulting fiber-reinforced solid. The derived constitutive relations incorporate direct microstructural information up to the two-point statistics. Since the resulting effective stored-energy function is realizable, in the sense that it is exact for a given class of microgeometries, it is guaranteed to be theoretically sound and to give physically sensible predictions. The predictions of the model are illustrated through stress-strain relations and loss of ellipticity criteria. The mechanical stability of fiber-reinforced Neo-Hookean solids is analyzed in the light of the new predictions.

1 INTRODUCTION

Soft solids that are reinforced by stiff cylindrical fibers constitute an important class of technological and biological material systems. The classical example is that of tires. Other prominent examples of more recent interest include nano-structured thermoplastic elastomers (Honecker *et al.*, 2000; Honecker and Thomas, 1996) and biological tissues such as arterial walls (Finlay *et al.*, 1998) and ligaments (Quapp and Weiss, 1998). Experimental evidence suggests that the macroscopic mechanical response of many among such classes of material systems is, to a first approximation, nonlinearly elastic. Making use of this simplifying assumption, there is a voluminous literature on *phenomenological* constitutive models for fiber-reinforced hyperelastic solids that are based on the theory of transversely isotropic invariants (see, for instance, Qui and Pence, 1997; Horgan and Saccomandi, 2005). More recently, constitutive models have been developed by means of the theory of *homogenization* (see, for instance, Lopez-Pamies and Ponte Castañeda, 2006, deBotton *et al.*, 2006; Agoras *et al.*, 2009), which depend not only on the properties of the constituents —*i.e.*, the matrix and the fibers— but also on the microstructure —*i.e.*, the size, shape, and orientation of the fibers.

In a recent effort, Lopez-Pamies and Idiart (submitted) have proposed a new homogenization-based constitutive theory for fiber-reinforced hyperelastic solids based on the idea of realizabitlity. The theory incorporates microstructural information in the form of one- and two-point probabilities, and is general enough to allow for any matrix and fiber stored-energy functions, and completely general loading conditions. Moreover, it has the distinguishing virtue of being *realizable*, in the sense that it reproduces exactly the behavior of material systems with a certain class of microgeometries. Consequently, the resulting effective stored-energy function is guaranteed to be objective, to satisfy all pertinent bounds, to linearize properly, to be exact to second order in the heterogeneity contrast, and to comply with any macroscopic constraints imposed by microscopic constraints, such as the strongly nonlinear constraint of incompressibility. The proposed formulation also grants access to information on the distribution of the local fields within each phase, which is required to characterize the evolution of microstructure and the onset of instabilities.

This constitutive theory is used in this paper to explore the dependence on microstructural variables of the elastic response and mechanical stability in fiber-reinforced Neo-Hookean solids. The Neo-Hookean model is commonly used to describe the mechanical response of rubbery materials at small and moderate deformations. It will be argued, however, that under certain loading conditions the resulting predictions remain valid for more general material models.

2 FIBER-REINFORCED NEO-HOOKEAN SOLIDS

2.1 Material model

We idealize a fiber-reinforced solid as a continuous matrix phase containing aligned cylindrical fibers that are *randomly* and *isotropically* distributed on the transverse plane, and are erfectly bonded to the matrix. The characteristic size of the cross section of the fibers is assumed to be much smaller than the size of the specimen and the scale of variation of the applied loads. It is further assumed that the random microstructure is statistically uniform and ergodic. The most important microstructural variables are the volume fraction and the orientation of the fibers in the *undeformed* configuration, denoted by c_0 and by a unit vector $\mathbf{N}$, respectively.

Let $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ be the deformation gradient, where $\mathbf{X}$ is the position vector of a material particle of the undeformed configuration and $\mathbf{x}$ is the corresponding position vector in the deformed configuration. The material systems considered here are incompressible and exhibit transversely isotropic symmetry. It is thus convenient to express their mechanical response in terms the four incompressible transversely isotropic invariants of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ about $\mathbf{N}$:

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}\left[(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}\mathbf{C}^2\right], \quad I_4 = \mathbf{N}\cdot\mathbf{C}\mathbf{N}, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\mathbf{N}. \tag{1}$$

Note that $I_3 = \det\mathbf{C} = 1$ in view of the postulated incompressibility.

Throughout the analysis, quantities associated with the *matrix* phase carry a superscript 1, while those associated with the *fiber* phase carry a superscript 2. The elastic response of the matrix material is characterized by an incompressible, *isotropic* Neo-Hookean stored-energy function

$$W^{(1)}(\mathbf{F}) = \frac{\mu^{(1)}}{2}(I_1 - 3), \tag{2}$$

where the material parameter $\mu^{(1)}$ represents a shear modulus in the ground state.

Because of their fabrication (or growth) process, fibers tend to be elastically anisotropic in such a way that the fiber direction represents a symmetry axis. Accordingly, the elastic response of the fibers is characterized here by an incompressible, *transversely isotropic* stored-energy function of the form

$$W^{(2)}(\mathbf{F}) = \frac{\mu_n^{(2)}}{2}(I_1 - 3) + g^{(2)}(I_4), \tag{3}$$

where the material parameter $\mu_n^{(2)}$ denotes a longitudinal shear modulus in the ground state, and $g^{(2)} : (0,\infty) \to [0,\infty)$ is a twice-differentiable *convex* function. The function $g^{(2)}$ is assumed to satisfy the standard conditions

$$g^{(2)}(1) = \frac{\mathrm{d}g^{(2)}}{\mathrm{d}I_4}(1) = 0, \quad \frac{\mathrm{d}^2 g^{(2)}}{\mathrm{d}I_4^2}(1) = \frac{3}{4}\left(\mu_a^{(2)} - \mu_n^{(2)}\right), \tag{4}$$

where the material constant $\mu_a^{(2)} \geq \mu_n^{(2)}$ corresponds to the axisymmetric shear modulus of the fibers in the ground state. An example of the form (3) commonly used in the literature is the so-called 'standard reinforcing model'

$$W^{(2)}(\mathbf{F}) = \frac{\mu_n^{(2)}}{2}(I_1 - 3) + \frac{3(\mu_a^{(2)} - \mu_n^{(2)})}{8}(I_4 - 1)^2. \tag{5}$$

2.2 The macroscopic stored-energy function

An estimate for the macroscopic stored-energy function of the above-described class of material systems can be obtained by means of the homogenization-based constitutive theory proposed by Lopez-Pamies and Idiart (submitted). The key point in this theory is to construct *random* fibrous microgeometries that permit the exact computation of the macroscopic properties of the resulting material systems. The construction process — which closely follows that of Idiart (2008) in the context of small-strain nonlinear elasticity — is comprised of two main steps. The first step consists of an iterated dilute homogenization procedure in finite elasticity that provides an exact result for the macroscopic stored-energy function of large classes of fiber-reinforced solids directly in terms of an auxiliary dilute problem. The second step deals with the formulation of the auxiliary dilute problem, which involves a novel class of sequential laminates whose matrix phase is present in dilute concentrations, and whose macroscopic response can be computed explicitly.

When applied to the above class of material systems, the resulting estimate for the macroscopic stored-energy function is given by (Lopez-Pamies and Idiart, submitted)

$$\overline{W}(\overline{\mathbf{F}}) = \frac{\widetilde{\mu}}{2}\left(\overline{I}_1 - 3\right) + \frac{\overline{\mu}_n - \widetilde{\mu}}{2}\frac{\left(\sqrt{\overline{I}_4}+2\right)\left(\sqrt{\overline{I}_4}-1\right)^2}{\sqrt{\overline{I}_4}} + c_0 g^{(2)}(\overline{I}_4), \qquad (6)$$

where

$$\overline{\mu}_n = (1-c_0)\mu^{(1)} + c_0\mu_n^{(2)} \quad \text{and} \quad \widetilde{\mu} = \frac{(1-c_0)\mu^{(1)} + (1+c_0)\mu_n^{(2)}}{(1+c_0)\mu^{(1)} + (1-c_0)\mu_n^{(2)}}\,\mu^{(1)} \qquad (7)$$

are two effective shear moduli.

It is worth noting that the stored-energy function (6) is bounded from above by the rigorous Voigt upper bound of Ogden (1978) for all $\overline{\mathbf{F}}$, as it should. When the macroscopic loading corresponds to aligned axisymmetric shear, the exact solution is known to be precisely the Voigt bound; the function (6) recovers such a result, as expected. It is also noted that for the case of isotropic fibers with $g^{(2)} = 0$, expression (6) reduces to an earlier estimate proposed by deBotton *et al.* (2006).

The stored-energy function (6) is expected to be reasonably accurate for Neo-Hookean solids reinforced by a transversely isotropic distribution of circular fibers with a very wide distribution of diameters, for the *entire range* of volume fractions $c_0 \in [0, 1]$. More specifically, for matrix-dominated modes of deformation —*e.g.*, transverse and longitudinal shear— the result (6) is expected to be accurate but somewhat soft —see comparisons with full-field simulations below. For fiber-dominated modes of deformation —*e.g.*, axisymmetric shear—, on the other hand, the result (6) is expected to be very accurate, in view of the fact that it reduces to the exact solution for aligned axisymmetric shear loadings.

2.3 Onset of instabilities

As the imposed level of deformation progresses beyond the linearly elastic neighborhood into the finite deformation regime, a fiber-reinforced solid may reach a point at which it becomes mechanically unstable. Mechanical instabilities are often times the precursors of failure in fiber-reinforced solids. They can be classified as ‘macroscopic’ instabilities, that is, geometric instabilities with wavelengths much larger than the characteristic size of the microstructure, and ‘local’ instabilities, which include geometric instabilities with wavelengths that are comparable to the characteristic size of the microstructure, as well as material instabilities — such as loss of strong ellipticity and cavitation — of the local constituents.

Lopez-Pamies and Idiart (submitted) noticed that the stored-energy functions (3) and (6) were of the separable form $W(\mathbf{F}) = \mathcal{F}(I_1) + \mathcal{G}(I_4)$. By considering the loss of strong ellipticity of that class of functions, they derived a closed-form criterion for the onset of instabilities of the localization-band type in such materials.

When applied to the fiber-reinforced Neo-Hookean solids of interest here, the following criterion results: along an arbitrary loading path with starting point $\overline{\mathbf{F}} = \mathbf{I}$, the stored-energy function (6) first becomes unstable at critical deformations $\overline{\mathbf{F}}_{cr}$ that satisfy

$$\begin{cases} \dfrac{\mu_n^{(2)}}{2} + \dfrac{\mathrm{d}g^{(2)}}{\mathrm{d}\overline{I}_4}(\overline{I}_4^{cr}) = 0 \\ \text{or} \\ \overline{I}_4^{cr} = \left[1 + \dfrac{2c_0}{\overline{\mu}_n}\dfrac{\mathrm{d}g^{(2)}}{\mathrm{d}\overline{I}_4}(\overline{I}_4^{cr})\right]^{-2/3}\left(1 - \dfrac{\widetilde{\mu}}{\overline{\mu}_n}\right)^{2/3} \end{cases} \tag{8}$$

where $\overline{I}_4^{cr} = |\overline{\mathbf{F}}_{cr}\mathbf{N}|^2$. Here, it is recalled that $\overline{\mu}_n$ and $\widetilde{\mu}$ are given, respectively, by expressions (7) in terms of the shear moduli $\mu^{(1)}$ and $\mu_n^{(2)}$ of the matrix and fiber phases, and of the volume fraction of fibers c_0.

Physically, when condition $(8)_1$ is satisfied, the fibers are prone to switch to a localized mode of (local) deformation. On the other hand, when condition $(8)_2$ is met, the fiber-reinforced solid is prone to switch to a localized mode of (macroscopic) deformation. Note that both types of instabilities can occur only when the deformation along the fiber direction, as measured by the invariant $\overline{I}_4$, is of a sufficiently large *compressive*[1] value $\overline{I}_4 = \overline{I}_4^{cr} \leq 1$ determined by certain ratios between the hard and soft modes of deformation. Indeed, the parameters $\mu_n^{(2)}$ and $\widetilde{\mu}$ characterize, respectively, the soft modes of deformation of the fibers and of the fiber-reinforced solid, while $\mathrm{d}g^{(2)}/\mathrm{d}\overline{I}_4$ and $\overline{\mu}_n$ characterize their hard modes.

From a computational point of view, it is also worth remarking that the nonlinear algebraic equations $(8)_1$ and $(8)_2$ cannot be solved explicitly for $\overline{I}_4^{cr}$ in general. For the special case when the fibers are taken to be Neo-Hookean (i.e., $g^{(2)} = 0$), however, $(8)_1$ is clearly never satisfied since $\mu_n^{(2)} > 0$ and equation $(8)_2$ leads to

$$\overline{I}_4^{cr} = \left(1 - \frac{\widetilde{\mu}}{\overline{\mu}_n}\right)^{2/3}. \tag{9}$$

This explicit result shows particulary well that the onset of instabilities is indeed controlled by the ratio of hard-to-soft modes of deformation (given here by $\widetilde{\mu}/\overline{\mu}_n$).

When the fibers are characterized by the standard reinforcing model (5), the criterion (8) can be written more explicitly as

$$\begin{cases} \overline{I}_4^{cr} = \dfrac{1 - (5/3)\delta_f}{1 - \delta_f} \\ \text{or} \\ (\overline{I}_4^{cr})^{3/2}\left[1 + \dfrac{3}{2}\dfrac{c_0(1-\delta_f)}{(1-c_0)\delta + c_0\delta_f}(\overline{I}_4^{cr} - 1)\right] = \dfrac{c_0(1-c_0)(\delta - \delta_f)^2}{[(1-c_0)\delta + c_0\delta_f][(1+c_0)\delta + (1-c_0)\delta_f]} \end{cases} \tag{10}$$

where the ratios

$$\delta = \frac{\mu^{(1)}}{\mu_a^{(2)}} \quad \text{and} \quad \delta_f = \frac{\mu_n^{(2)}}{\mu_a^{(2)}} \tag{11}$$

serve to measure, respectively, the heterogeneity contrast between the matrix and the fibers (in the fiber direction), and the fiber anisotropy[2]. While $(10)_1$ is explicit, condition $(10)_2$ is

[1] Recall that $\mathrm{d}g^{(2)}(\overline{I}_4^{cr})/\mathrm{d}\overline{I}_4 > 0 (< 0)$ for $\overline{I}_4^{cr} > 1 (< 1)$ and $\overline{\mu}_n \geq \widetilde{\mu} > 0$, so that $\overline{I}_4^{cr}$ in (8) is necessarily less than or equal to 1.

[2] Fibers are typically stiffer in the axial direction than in the transverse direction, so that $\delta_f < 1$.

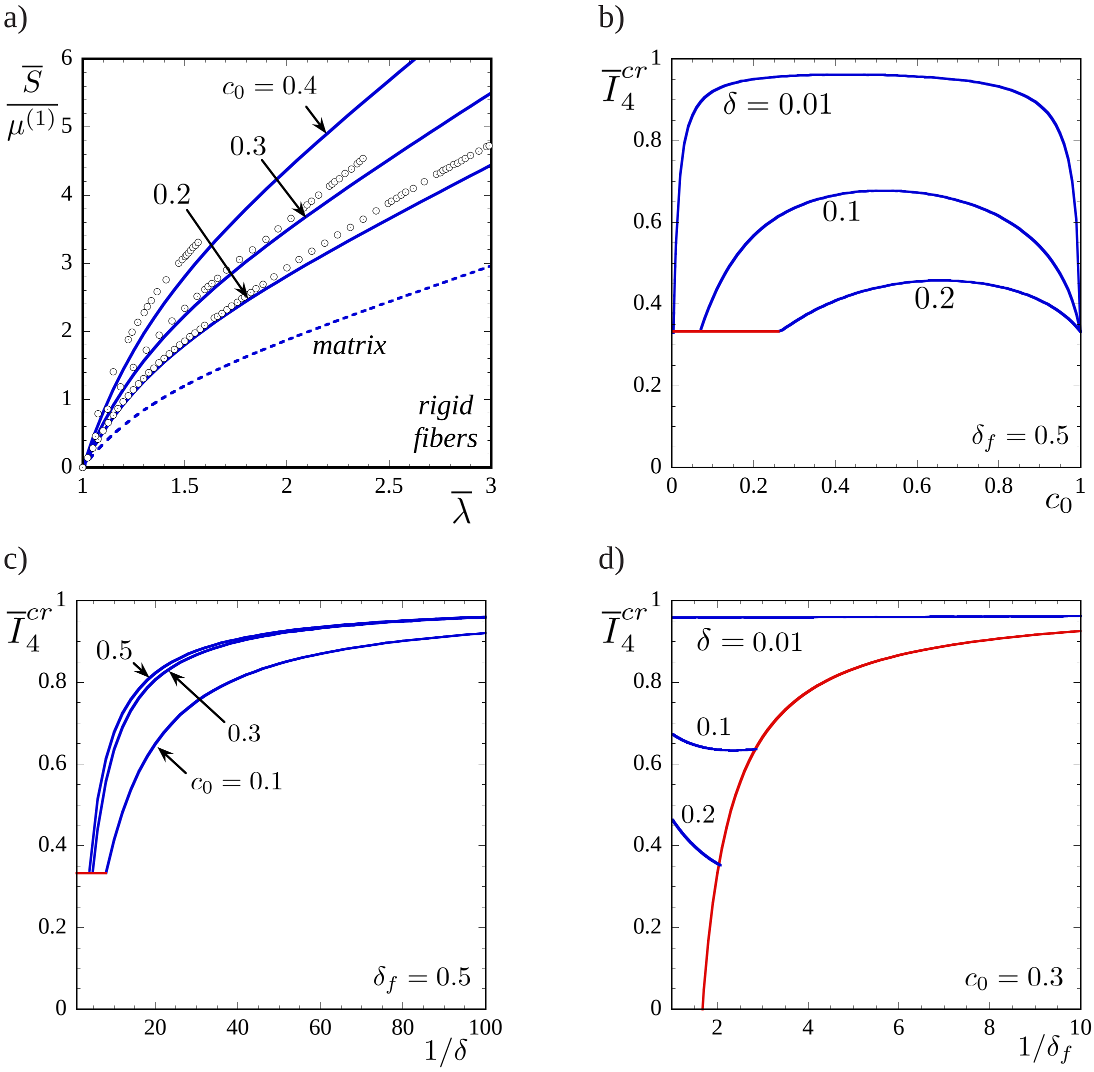

Figure 1: a) Stress-stretch response of a Neo-Hookean solid reinforced with rigid fibers, subjected to in-plane pure shear loading. (b-d) Critical deformation for the onset of instabilities: a) as a function of fiber concentration c_0, for one value of fiber anisotropy parameter ($\delta_f = 0.5$) and various values of the heterogeneity contrast ($\delta = 0.01, 0.1, 0.2$), b) as a function of the heterogeneity contrast δ, for one value of fiber anisotropy parameter ($\delta_f = 0.5$) and various values of the fiber concentration ($c_0 = 0.1, 0.3, 0.5$), c) as a function of fiber anisotropy parameter δ_f, for one value of fiber concentration ($c_0 = 0.3$) and various values of the heterogeneity contrast ($\delta = 0.01, 0.1, 0.2$). Blue lines are associated with macroscopic instabilities, while red lines are associated with fiber instabilities.

a nonlinear algebraic equation for $\overline{I}_4^{cr}$ that must be solved numerically. If the fibers are much stiffer than the matrix, however, the ratio δ is very small, and condition (10)$_2$ leads to the explicit asymptotic result

$$\overline{I}_4^{cr} = 1 - \frac{2(1+c_0)}{3c_0(1-c_0)}\,\delta + O(\delta^2). \tag{12}$$

3 RESULTS AND DISCUSSION

The above formulation is used here to explore the effect of the various microstructural parameters on the macroscopic response and mechanical stability of fiber-reinforced Neo-Hookean solids with anisotropic fibers with stored-energy function (5).

Figure 1a shows stress-stretch curves for a Neo-Hookean solid reinforced with rigid fibers, subjected to in-plane pure shear deformations[3]. The new predictions are compared with full-field numerical simulations of Neo-Hookean solids reinforced with a random and isotropic distribution of monodisperse circular fibers (Moraleda et al., 2009). Good agreement is found for the three levels of reinforcement concentration ($c_0 = 0.2, 0.3, 0.4$) and the entire range of deformations considered. The fact that the predictions are consistently lower than the simulations is in agreement with the observation that the above constitutive model should be appropriate for material systems with polydisperse fiber size distributions. In any event, the accuracy of the predictions is remarkable given that the numerical simulations reveal a severe evolution of the microstructure and a strongly heterogeneous distribution of the mechanical fields within the matrix phase with increasing stretch (see fig. 6 in Moraleda et al., 2009).

Figures 1b-d show the effect of the various material parameters on the critical deformation for the onset of instabilities in fiber-reinforced Neo-Hookean solids. In these figures, blue and red lines correspond, respectively, to the loss of ellipticity of the macroscopic and fiber stored-energy functions, as characterized, respectively, by conditions $(8)_2$ and $(8)_1$. The effect of reinforcement concentration c_0 and heterogeneity contrast δ is shown in figs. 1b & c, at a particular value of fiber anisotropy ($\delta_f = 0.5$). It is recalled that the Neo-Hookean stored-energy function is strongly elliptic. As soon as fibers are added, however, the effective stored-energy function loses strong ellipticity at some finite deformation level. Predictions show that fiber-reinforced solids are progressively more unstable —i.e., $\overline{I}_4^{cr}$ tends to 1— with increasing contrast —i.e., decreasing δ, as expected. By contrast, the effect of the fiber concentration is not monotonic. At small c_0, the composite is most stable and fails by loss of ellipticity of the fiber phase. As the fiber concentration increases, the failure mode switches to macroscopic loss of ellipticity and the composite becomes progressively more unstable up to a certain value of c_0 after which the trend reverses. Thus, the model predicts that there are certain values of c_0 at which fiber-reinforced solids are most unstable, and that the fiber-reinforced solid cannot be more stable than the fiber phase.

Finally, the effect of fiber anisotropy is considered in fig. 1d. Recall that the constitutive model adopted for the fibers reduces to the Neo-Hookean model when the fibers are isotropic ($\delta_f = 1$). In this case, both the matrix and the fiber phases are strongly elliptic, but the effective stored-energy function predicted by the model still loses strong ellipticity at a finite value of $\overline{I}_4^{cr}$ due to the rotation of the fibers.

4 CONCLUDING REMARKS

More often than not, the fibers in these material systems are much stiffer than the matrix. In that case, the above constitutive model predicts an $\overline{I}_4^{cr}$ close to 1, given by the asymptotic result (9). As already argued by Agoras et al. (2009), a practical implication of this result is that, when considering the compressive failure in fiber-reinforced nonlinearly elastic materials with very stiff fibers, it suffices to model the matrix phase as Neo-Hookean.

Finally, we note that expression (8) for $\overline{I}_4^{cr}$ describes an onset-of-failure surface in $\overline{\mathbf{F}}$-space. This surface accounts for two failure mechanisms: local fiber instabilities and macroscopic — or long wavelength — instabilities. However, other local mechanisms that are precursors of failure can also be incorporated to the proposed formulation. Two notable examples are fiber debonding and matrix cavitation. Failure surfaces that account for all these mechanisms simultaneously are of great practical importance and efforts to determine them are currently

[3] $\overline{\mathbf{F}} - \mathbf{I} = (\overline{\lambda} - 1)\mathbf{e}_1 \otimes \mathbf{e}_1 + (\overline{\lambda}^{-1} - 1)\mathbf{e}_2 \otimes \mathbf{e}_2$, where $\mathbf{e}_1 \cdot \mathbf{N} = \mathbf{e}_2 \cdot \mathbf{N} = 0$. The stress is given by $\overline{S} = \hat{W}'(\overline{\lambda})$, where $\hat{W}(\overline{\lambda}) = \overline{W}(\overline{\mathbf{F}})$.

under way.

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