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DYNAMIC DIFFUSION FORMULATION FOR ADVECTION DOMINATED TRANSPORT PROBLEMS

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Abstract. Two discontinuous dynamic diffusion formulations for the numerical solution of advection-diffusion equations are proposed in this work. The first one reformulates, using broken spaces, the Consistent Approximate Upwind Petrov-Galerkin (CAU) finite element model. The second one considers a two-scale framework and introduces an artificial diffusion to DG formulation that acts isotropically in both scales. The amount of artificial diffusion is dynamically determined by the resolved scale solution at an element level, yielding a self adaptive and parameter-free method. This formulation takes into account the effective flux through inter-element edges to keep the consistency property. Numerical experiments are conducted, which cover a variety of problem parameter ranges, in order to show the behavior of the proposed methods in comparison with some discontinuous Galerkin methods.

1 INTRODUCTION

The design of numerical methods for advection dominant transport equations aims to yield stability and accuracy. As far as finite element methods are concerned, the most common approach to reach this aim is known as stabilized methods. These methods add to the Galerkin formulation perturbation terms associated to the operator and containing stabilizing parameters dependent on the mesh. In essence, they add some sort of artificial dissipation to avoid energy accumulation at the grid scale. The accuracy and stability of the solutions obtained with these methodologies rely on suitable designs of the stabilization parameter(s) (John and Knobloch (2007, 2008)). These works present a review and comparison among many conforming methods which have been proposed in the literature to overcome spurious oscillations. An interesting approach to evaluate the parameters in numerical methods was developed in Oberai and Wanderer (2005a), derived from an extension of the variational Germano identity (Oberai and Wanderer (2005b)). In this approach, some restrictions of the numerical solution onto coarse function spaces are imposed and the parameters are easily computed if they are assumed to be constant in the computational domain. The obtained parameters typically depend on the approximate solution, yielding nonlinear methods.

The Germano identity, initially derived in Germano et al. (1991), is a popular tool for computing the magnitude of the eddy viscosity in the large eddy simulation of turbulent flows. The dynamic introduction of some artificial dissipation (eddy viscosity) is the basis of this procedure, based on the notion of scale separation. The idea of applying eddy viscosity models only onto the small scales results in the subgrid eddy viscosity method developed in Guermond (2001); Kaya and Rivière (2005), which is similar in spirit to the spectral viscosity technique introduced by Tadmor (1989) to approximate nonlinear conservation equations by means of spectral methods (Ern and Guermond (2004)). The methods proposed in Guermond (2001); Kaya and Rivière (2005) still require tunable parameters whose selection is a tricky task for actual problems. The nonlinear subgrid diffusivity methods presented in Santos and Almeida (2007); Arruda et al. (2010) were attempts to avoid user-defined coefficients by means of a subgrid diffusivity which is a nonlinear local functional of the resolved scale solution. These free parameter methods present good stability and convergence properties for singular perturbed transport problems, although oscillations still remain in some situations, mainly when the velocity field is not constant and when external layers are present. In this latter case, appearing in conforming formulations, spurious modes may only be overcome by relaxing the boundary conditions in a DG way.

Motivated by eddy viscosity models in which the dissipation mechanism is introduced either on all scales or on the subgrid scale, we propose a discontinuous two-scale method where an artificial diffusion appears on all scales. The additional diffusion is dynamically determined by imposing some restrictions on the resolved scale solution in the same spirit of the methods presented in Santos and Almeida (2007); Arruda et al. (2010), which assume a two level decomposition of the velocity field. This approach results in a free parameter method, called Dynamic Diffusion method (DD), in which an artificial diffusion model acts isotropically and is locally adapted to guarantee stability of the resolved scale solution. We find that the proposed methodology outperforms some subgrid diffusion approaches for transport problems as well as some discontinuous capturing methods. In particular, we reformulate, using broken spaces, the Consistent Approximate Upwind Petrov-Galerkin (CAU) finite element model and we compared it with the new proposed dynamic diffusion discontinuous method.

The outline of this paper is as follows. Section 2 briefly addresses the transport problem and introduces the used notation. In section 3, the discontinuous version of the CAU method is presented. The Dynamic Diffusion method is introduced in Section 4. Numerical experiments are conducted in Section 5 to show the behavior of the proposed methodologies for a variety of transport problems as well as convergence rates for regular solutions. Section 6 concludes this paper.

2 PROBLEM STATEMENT AND NOTATION

We consider the steady-state advection-diffusion-reaction equation as follows:

$$\begin{aligned} \mathcal{L}u &= -\varepsilon\Delta u + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} u + \sigma u = f, \quad \text{in } \Omega; \\ u &= g \quad \text{on} \quad \Gamma_d, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2)$ is an open bounded domain with a Lipschitz boundary Γ. We define the inflow and outflow parts of Γ, respectively in the usual fashion:

$$\Gamma_- = \{x \in \Gamma : \boldsymbol{\beta}(x)\cdot\boldsymbol{n}(x) < 0\}; \tag{2}$$

$$\Gamma_+ = \{x \in \Gamma : \boldsymbol{\beta}(x)\cdot\boldsymbol{n}(x) > 0\}, \tag{3}$$

where $\boldsymbol{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$. In (1), Δ and $\boldsymbol{\nabla}$ are the Laplacian and gradient operators, respectively; $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d$ is the divergence free velocity field; ε is the diffusion coefficient; $g \in H^{1/2}(\Gamma_d)$; $\sigma \in L_\infty(\Omega)$ and $f \in L_2(\Omega)$, the reaction coefficient and the source term, respectively, are real-valued functions. We also assume that there exists a real constant σ_0 such that $\sigma > \sigma_0 \geq 0$. We define $\Theta := \{\boldsymbol{\varepsilon}, \boldsymbol{\beta}, f\}$ as the set of input data. Moreover, for functions w and v in $(L_2(D))^m$, $D \subset \mathbb{R}^d$ and $m \geq 1$, let $(w,v)_D = \int_D w\cdot v \, d\boldsymbol{x}$. For functions w and v in $(L_2(\Upsilon))^m$, $\Upsilon \subset \mathbb{R}^{d-1}$, $\langle w,v\rangle_\Upsilon = \int_\Upsilon w\cdot v \, d\boldsymbol{s}$. Also, hereafter $H^1(\cdot)$ and $H^1_0(\cdot)$ denote the usual Sobolev spaces.

In order to derive a DG formulation for (1), we have to introduce some notation. The subgrid stabilization considered here is based on a two-level discretization so that two nested grids must be built. We consider a coarsest regular triangulation $\mathcal{T}_H$ of the domain Ω into triangles T_H, where H stands for the diameter of T_H in $\mathcal{T}_H$. For each triangle $T_H \in \mathcal{T}_H$, four triangles are created by connecting the midpoint of the edges and the resulting finer triangulation is denoted by $\mathcal{T}_h$. Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}_h$. Let $e_M = \{e_1, e_2\}$, $e_j \in \mathcal{E}_h$, $j = 1,2$, be an edge of a macro triangle $T_H \in \mathcal{T}_H$. Figure 1 shows the particular case when e_M is an interior edge shared by the macro triangles $T_1, T_2 \in \mathcal{T}_H$ and $e_j \in e_M$ is shared by triangles $T_{1,j}, T_{2,j} \in \mathcal{T}_h$. The set of all edges of $\mathcal{T}_H$ is then defined by $\mathcal{E}_H = \cup_{\mathcal{T}_H} e_M$. We also define $\mathcal{E} = \mathcal{E}_h \cap \mathcal{E}_H = \mathcal{E}^0 \cup \mathcal{E}^\Gamma$, where $\mathcal{E}^0$ and $\mathcal{E}^\Gamma$ are the sets of internal edges and of edges on the boundary Γ, respectively. The generic edge e that is in an inflow part of the domain belongs to the set $\mathcal{E}^{0-} = \{e \in \mathcal{E}^0 : \boldsymbol{\beta}(x)\cdot\boldsymbol{n}(x) < 0, \forall x \in e\}$, if it is an interior edge, or to $\mathcal{E}_h^{\Gamma-} = \mathcal{E}^\Gamma \cap \Gamma_-$, if it lies on the boundary. Moreover, let $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ be the unit normal vectors on e_j pointing outward to $T_{1,j}$ and $T_{2,j}$, respectively. We introduce boundary operators as averages and jumps of scalar and vector-valued functions φ and $\boldsymbol{\tau}$ in the following way: piecewise smooth on $\mathcal{T}_h$, with $\varphi^i := \varphi|_{T_{i,j}}$ for each edge e_j, $j = 1,2$, we define

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\varphi]\!] = \varphi^1\boldsymbol{n}^1 + \varphi^2\boldsymbol{n}^2 \quad \text{for all internal edges;} \tag{4}$$

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^1\cdot\boldsymbol{n}^1 + \boldsymbol{\tau}^2\cdot\boldsymbol{n}^2 \quad \text{for all internal edges,} \tag{5}$$

with $\varphi^i := \varphi|_{T_{i,j}}$ and $\boldsymbol{\tau}^i := \boldsymbol{\tau}|_{T_{i,j}}$. For all edges on the boundary Γ, the jump and average operators of φ and $\boldsymbol{\tau}$ coincide with their traces on e.

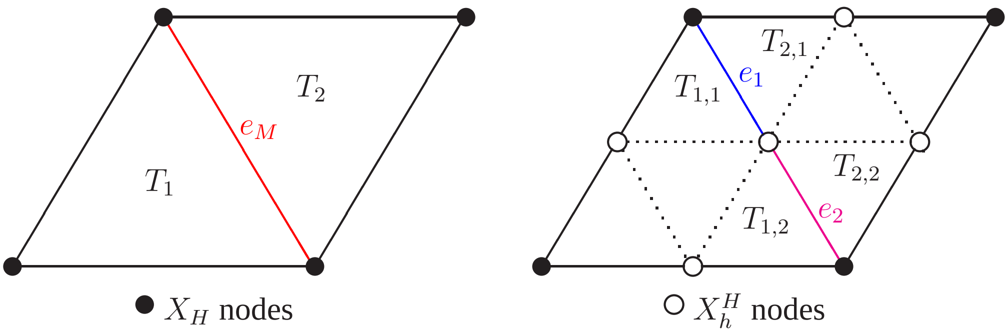


Figure 1: Schematic representation of the *two-level* $\mathbb{P}_1$ *setting*.

A two-level piecewise linear finite element approximation is defined by introducing the following two broken spaces on the partition $\mathcal{T}_H$:

$$X_H = \{u_H \in L_2(\Omega) \mid u_H|_{T_H} \in H^1(T_H) \cap \mathbb{P}_k(T_H),\ \forall T_H \in \mathcal{T}_H\}; \tag{6}$$

$$X_h = \{u_h \in L^2(\Omega) \mid u_h|_{T_H} \in H^1(T_H),\ u_h|_{T_h} \in \mathbb{P}_k(T),\ \forall T \in \mathcal{T}_h, \forall T_H \in \mathcal{T}_H\}, \tag{7}$$

where $\mathbb{P}_k$ stands for the set of interpolation polynomials of degree less or equal to k. In this paper we set $k = 1$ and we denote the finite element approximation defined by the the couple (X_H, X_h) by two-level $\mathbb{P}_1$. Higher order polynomials may be used as well. We also introduce an additional discrete space $X_h^H \subset X_h$, such that the following decomposition holds:

$$X_h = X_H \oplus X_h^H, \tag{8}$$

where X_H is the resolved (coarse) scale space whereas X_h^H is the subgrid (refined) scale space. Given $u_h \in X_h$ and $u_H \in X_H$ such that u_h and u_H coincide in the coarse scale nodes, we define $u_h^H \in X_h^H$ and the space decomposition (8) implies that $u_h^H = u_h - u_H$. More precisely, let $P_H : X_h \to X_H$ be the projection of X_h onto X_H that is parallel to X_h^H. For all $v_h \in X_h$ we set $v_H = P_H v_h$ and $v_h^H = (I - P_H)\, v_h$. The definition of P_H for the two-level $\mathbb{P}_1$ setting defined by the couple (X_H, X_h) is given in Ern and Guermond (2004) (page 242).

One may notice that the space X_h is required to be continuous inside of each T_H, although it is discontinuous along $e_M \subset \mathcal{E}_H$. As a two-level piecewise linear finite element approximation will be considered here, the subgrid scale solution may be nonzero across the $\mathcal{T}_H$ mesh lines. This property will be used to build the new nonlinear discontinuous subgrid formulation in section 4.

A DG (Discontinuous Galerkin) formulation for (1) is given by: find $u_h \in X_h$ such that

$$B_{DG}(u_h, v_h) = F_{DG}(v_h), \quad \forall v_h \in X_h, \tag{9}$$

where

$$\begin{aligned} B_{DG}(u_h, v_h) &= \sum_{T\in\mathcal{T}_h} (\varepsilon \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v_h)_T + \sum_{T\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h + \sigma u_h, v_h)_T \\ &+ \sum_{e\in\mathcal{E}_h^0} \langle [\![u_h]\!], \{\varepsilon\nabla v_h\}\rangle_e - \sum_{e\in\mathcal{E}_h^0} \langle \{\varepsilon\nabla u_h\}, [\![v_h]\!]\rangle_e + \sum_{e\in\mathcal{E}_h^0} \langle \eta_1 [\![u_h]\!]\cdot[\![v_h]\!]\rangle_e \\ &- \sum_{e\in\mathcal{E}_h^{0-}} \langle \boldsymbol{\beta}\cdot[\![u_h]\!], v_h\rangle_e - \sum_{e\in\mathcal{E}_h^{\Gamma-}} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n}) u_h, v_h\rangle_e \\ &+ \sum_{e\in\mathcal{E}_h^{\Gamma}} \langle u_h, (\varepsilon\boldsymbol{\nabla} v_h\cdot\boldsymbol{n})\rangle_e - \sum_{e\in\mathcal{E}_h^{\Gamma}} \langle (\varepsilon\boldsymbol{\nabla} u_h\cdot\boldsymbol{n}), v_h\rangle_e + \sum_{e\in\mathcal{E}_h^{\Gamma}} \langle \eta_1 u_h, v_h\rangle_e \;; \end{aligned} \quad (10)$$

$$F_{DG}(v_h) = \sum_{T\in\mathcal{T}_h} (f, v_h)_T + \sum_{e\in\mathcal{E}_h^{\Gamma}} \langle g, (\varepsilon\boldsymbol{\nabla} v_h\cdot\boldsymbol{n})\rangle_e + \sum_{e\in\mathcal{E}_h^{\Gamma}} \langle \eta_1 g, v_h\rangle_e - \sum_{e\in\mathcal{E}_h^{\Gamma-}} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n}) g, v_h\rangle_e \,, \quad (11)$$

and $\eta_1 = 4\varepsilon/s$, $s = \min\{meas(T_{1,j})^{1/d}, meas(T_{2,j})^{1/d}\}$, $j \in \{1,2\}$, where $T_{1,j}, T_{2,j} \in \mathcal{T}_h$ are the triangles sharing an edge e.

It is well known that formulation (9) might present non-physical oscillations in the neighborhood of steep gradients, so that some additional stabilization may be necessary, depending on the problem. Some approaches to overcome this difficulty are based on slope limiters (Biswas et al. (1994); Hoteit et al. (2004); Klieber and Rivière (2006)), Petrov-Galerkin stabilizations (Houston et al. (2000); Brezzi et al. (2006))], bubble stabilization (Antonietti et al. (2009); Rochinha et al. (2007)), interior penalty-type stabilizations (Brezzi et al. (2006); Burman (2005)) and subgrid stabilization (Kaya and Rivière (2005)). In essence, the last three approaches use some sort of artificial diffusion to improve stability, which will be used here by considering a discontinuity-capturing and a subgrid formulation. As we will see in the following, although having much in common, they present remarkable differences that yield different accuracy behaviors.

3 DCAU - DISCONTINUOUS CAU METHOD

The stability of the DG formulation (9) may be generically improved by adding a model term such that the formulation reads:

$$\begin{cases} \text{Find } u_h \in X_h \text{ such that} \\ B_{DG}(u_h, v_h) + \mathcal{M}(u_h, v_h; \Theta, h, \tau) = F_{DG}(v_h), \quad \forall v_h \in X_h \,. \end{cases} \quad (12)$$

In other words, the model term $\mathcal{M}(\cdot,\cdot;\cdot,\cdot,\cdot)$ depends on the weighting function v_h, the trial function u_h, may depend on the input data and the characteristic mesh size h. Eventually, it may also depend on one or more user-specified non-negative parameter $\tau \in L^\infty(\Omega)$. Different model terms yield different methods. The SUPG method, for example, provides an additional control of the solution gradient in the streamline direction. It is originally proposed in Brooks and Hughes (1982), whose model term is

$$\mathcal{M}_{supg} = \sum_{T\in\mathcal{T}_h} (\mathcal{L}u_h - f, \tau_{supg}\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v_h)_T \,, \quad (13)$$

was first combined with DG formulations in Johnson et al. (1984) for linear hyperbolic problems. The discontinuous method proposed in Houston et al. (2000) combines the DG formulation (9) with (13), which yields quasi-optimal error estimates in a mesh-dependent norm. A gap of $1/2$ is observed in the L_2 error estimates for regular solutions. However, as its continuous counterpart, spurious modes might still appear near steep gradients. A common approach to overcome or to reduce this difficulty is to use a discontinuity-capturing model (see John and Knobloch (2007) for a review). Many discontinuity-capturing models are equivalent to split the stabilization model term into two parts. One part is a linear stabilization term, like the standard SUPG operator (13). The second part is intended to control the solution gradient on the remaining directions, preventing localized oscillations around boundary layers. It usually depends on the approximate solution, hence it introduces a nonlinearity in the solution process. A method of this type is the Consistent Approximate Upwind Petrov-Galerkin (CAU) finite element model proposed in Galeão and do Carmo (1988). In this method, the model term is given by

$$\mathcal{M}_{cau} = \sum_{T \in \mathcal{T}_h} (\mathcal{L}u_h - f, \tau_{supg} \boldsymbol{\beta} \cdot \boldsymbol{\nabla} v_h)_T + \sum_{T \in \mathcal{T}_h} \left(\tau_{cau} \frac{|\mathcal{L}u_h - f|^2}{|\boldsymbol{\nabla} u_h|^2} \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v_h \right). \tag{14}$$

This term is obtained from the definition of an approximate upwind direction $\boldsymbol{\beta}_{app} = \tau_1 \boldsymbol{\beta} + (\tau_2 - \tau_1) \left(\boldsymbol{\beta} - \overline{\boldsymbol{\beta}} \right)$ that considers an auxiliary velocity field $\overline{\boldsymbol{\beta}}$, which is obtained by requiring that it must be as close as possible from the actual velocity field (in the L_2 sense) and $-\varepsilon \Delta u_h + \overline{\boldsymbol{\beta}} \cdot \boldsymbol{\nabla} u_h + \sigma u_h - f = 0$ in each $T \in \mathcal{T}_h$. These conditions lead to

$$\boldsymbol{\beta} - \overline{\boldsymbol{\beta}} = \frac{\mathcal{L}u_h - f}{|\boldsymbol{\nabla} u_h|^2} \boldsymbol{\nabla} u_h \quad \text{if } |\boldsymbol{\nabla} u_h| \neq 0,$$

and $\boldsymbol{\beta} - \overline{\boldsymbol{\beta}} = 0$ otherwise. Thus, the model term $\mathcal{M}_{cau}$ is obtained by taking $\boldsymbol{\beta}_{app} \cdot \boldsymbol{\nabla} v_h$ as a Petrov-Galerkin perturbation of the weight function inside each element $T \in \mathcal{T}_h$, assuming $\tau_1 = \tau_{supg}$ and $\tau_2 - \tau_1 = \tau_{cau}$. We may notice that the formulation (12) with (14) preserves consistency property of the DG method and the stability is enhanced by introducing the artificial diffusion associated to (14). However, the correct choice of the stabilization parameters (τ_{supg} and τ_{cau} in this case) is a key issue here and in many stabilization procedures as well. In the numerical experiments we perform here, we use the definition proposed in Almeida and Silva (1997).

We should remark that a shock capturing procedure may also be introduced without including a linear stabilization term. This is the approach followed in Hartmann (2006), in which an interior penalty DG method with the model term $\mathcal{M} = \sum_{T \in \mathcal{T}_h} (\epsilon(u_h) \boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v_h)$ is used to reduce overshoots at discontinuities for the compressible Navier-Stokes equations.

4 DD- DYNAMIC DIFFUSION METHOD

There are many multiscale methods that can be put into the general form (12). In a conforming setting, the linear subgrid scale method SGS developed in Guermond (1999) defines the model term as

$$\mathcal{M}_{SGS} = \tau_{sgs} \sum_{T \in \mathcal{T}_h} \left(\boldsymbol{\nabla} u_h^H, \boldsymbol{\nabla} v_h^H \right)_T,$$

which amounts to introduce a dissipative operator associated only to the subgrid scales. The SGS main drawback is that the amount of artificial diffusion depends on the choice of τ_{sgs}, a (global) user-defined parameter that plays a crucial role in the accuracy of the method. This issue drove the development of the conforming nonlinear subgrid scale method (NSGS) for advection-diffusion problems presented in Santos and Almeida (2007), whose the main idea is to decompose the velocity field into the resolved (coarse) and unresolved (subgrid) scales, with respect to the grid scales, as $\boldsymbol{\beta} = \boldsymbol{\beta}_H + \boldsymbol{\beta}_h^H$. Inspired by the CAU method, the subgrid velocity field $\boldsymbol{\beta}_h^H$ is determined by requiring the minimum of the associated kinetic energy ($\frac{1}{2}\left|\boldsymbol{\beta}_h^H\right|^2$) for which the residual of the resolved scale solution (with $\boldsymbol{\beta}_H$) vanishes. The solution of this minimization problem yields a subgrid velocity field at each finite element that is effectively a projection of the coarse scale residual along the gradient of the resolved scale solution when $\boldsymbol{\nabla} u_H \neq \mathbf{0}$ in the form:

$$\boldsymbol{\beta}_h^H = \frac{R(u_H)}{|\boldsymbol{\nabla} u_H|^2}\boldsymbol{\nabla} u_H, \tag{15}$$

where $R(u_H) = -\varepsilon\Delta u_H + \boldsymbol{\beta}\cdot\nabla u_H + \sigma u_H - f$. When $\boldsymbol{\nabla} u_H = \mathbf{0}$, the subgrid velocity vanishes. Thus, the amount of subgrid diffusion required to dissipate the small scale kinetic energy is defined as

$$\varepsilon_h^H := \frac{1}{2} h\left|\boldsymbol{\beta}_h^H\right| \tag{16}$$

and the NSGS model term is given by

$$\mathcal{M}_{NSGS} = \sum_{T\in\mathcal{T}_h}\left(\varepsilon_h^H \boldsymbol{\nabla} u_h^H, \boldsymbol{\nabla} v_h^H\right)_T \quad \text{if } |\boldsymbol{\nabla} u_H| \neq \mathbf{0},$$

and $\mathcal{M}_{NSGS} = 0$ otherwise. It is a free parameter method since the the artificial diffusion ε_h^H is not determined a priori but is evaluated based on the resolved scale approximate solution u_H.

In Arruda et al. (2010) the NSGS was reformulated using broken spaces, which are defined on the coarsest partition. The approximation space X_h, defined on $\mathcal{T}_h$, is split into resolved scales (X_H) and subgrid scales (X_H^h) spaces so that continuity is enforced inside each macro element. The formulation, called NSDG, is built by considering the following subgrid eddy viscosity model, *à la* NSGS, and an additional subgrid edge stabilization

$$\mathcal{M}_{NSDG} = \sum_{T\in\mathcal{T}_h}\left(\varepsilon_h^H \boldsymbol{\nabla} u_h^H, \boldsymbol{\nabla} v_h^H\right) + \sum_{e\in\mathcal{E}^0}\int_e \eta_2 [\![u_h^H]\!]\cdot[\![v_h^H]\!] dS, \tag{17}$$

where $\eta_2 = 4\varepsilon_h^H/s$. The first term of (17) acts inside each element of $\mathcal{T}_H$ (or $\mathcal{T}_h$), introducing the necessary amount of artificial diffusion to dissipate the kinetic energy associated to the unresolved scales. The second term, weighted by η_2, introduces a penalty of the unresolved scale jumps. Since η_2 depends on ε_h^H, which depends on the residual of the resolved scale solution inside each $T_H \in \mathcal{T}_H$ ($R(u_H)|_{T_H}$), this penalty term is also self adaptive. Optimal convergence rates are obtained for regular problems. However, oscillations still remain in some situations, mainly when the velocity field is not constant.

A new version of a two-scale method is introduced here aiming to solve these drawbacks. The underline idea still is to control the resolved scale solution so that the spurious modes are confined to the subgrid scales. We call this method the dynamic diffusion (DD) method since it adds to the DG formulation (9) the following model term:

$$\mathcal{M}_{DD} = \sum_{T\in\mathcal{T}_h} \xi_h^H|_{T_H}\left(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} u_h\right)_T, \tag{18}$$

which dynamically introduces an isotropic artificial diffusion onto all scales. The consistency, stability and convergence properties of the resulting methodology relies on the definition of the artificial diffusion $\xi_h^H|_{T_H}$. Like in the NSGS and NSDG methods, its magnitude is dynamically determined by imposing the same restrictions on the resolved scale solution. However, instead of (16), $\xi_h^H|_{T_H}$ is assumed constant in each $T_H \in \mathcal{T}_H$ and determined as

$$\xi_h^H|_{T_H} = \frac{1}{2}\hbar \left\langle |\boldsymbol{\beta}_h^H| \right\rangle, \quad \text{where } \left\langle |\boldsymbol{\beta}_h^H| \right\rangle = \frac{\int_{T_H} \left|\boldsymbol{\beta}_h^H\right| d\boldsymbol{x}}{\int_{T_H} d\boldsymbol{x}} \tag{19}$$

and the length scale $\hbar$ is defined as

$$\hbar = \frac{\left|\boldsymbol{\beta}_h^H\right|}{\left|\boldsymbol{b}_{T_H}\right|}, \quad \boldsymbol{b}_{T_H} = \left(\boldsymbol{\nabla}_{N_{T_H}}\right)\boldsymbol{\beta}_h^H,$$

where the tensor $\boldsymbol{\nabla}_{N_{T_H}}$ is the Jacobian of the coordinate system and N_{T_H} denotes the element local coordinates. With these definitions, we generalize the measure of the subgrid velocity field $\left\langle |\boldsymbol{\beta}_h^H| \right\rangle$ for any space dimension and the length scale $\hbar$ takes into account the direction of the subgrid velocity field $\boldsymbol{\beta}_h^H$, decreasing the dependence of the solution with the mesh orientation.

Another important issue in the design of this new method is related to the effective flux through inter-element edges to keep the consistency property. In order to keep optimal convergence order in the L_2 norm, the DG formulation must consider the effective diffusion introduced by the model. To this end, let $\zeta = \varepsilon + \xi_h^H$ denote the effective diffusion coefficient that acts inside each macro element T_H. Thus, the DD method reads: find $u_h \in X_h$ such that

$$B_{DD}(u_h, v_h) + \mathcal{M}_{DD} = F_{DD}(v_h), \quad \forall v_h \in X_h, \tag{20}$$

where

$$\begin{aligned} B_{DD}(u_h, v_h) \;=\; & \sum_{T\in\mathcal{T}_h} (\varepsilon\boldsymbol{\nabla} u_h, \boldsymbol{\nabla} v_h)_T + \sum_{T\in\mathcal{T}_h} (\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h + \sigma u_h, v_h)_T \\ & + \sum_{e\in\mathcal{E}^0} \langle [\![u_h]\!], \{\zeta\boldsymbol{\nabla} v_h\}\rangle_e - \sum_{e\in\mathcal{E}^0} \langle \{\zeta\boldsymbol{\nabla} u_h\}, [\![v_h]\!]\rangle_e + \sum_{e\in\mathcal{E}^0} \langle \overline{\eta}_1 [\![u_h]\!]\cdot[\![v_h]\!]\rangle_e \\ & - \sum_{e\in\mathcal{E}^{0-}} \langle \boldsymbol{\beta}\cdot[\![u_h]\!], v_h\rangle_e - \sum_{e\in\mathcal{E}^{\Gamma-}} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n}) u_h, v_h\rangle_e \\ & + \sum_{e\in\mathcal{E}^\Gamma} \langle u_h, (\zeta\boldsymbol{\nabla} v_h\cdot\boldsymbol{n})\rangle_e - \sum_{e\in\mathcal{E}^\Gamma} \langle (\zeta\boldsymbol{\nabla} u_h\cdot\boldsymbol{n}), v_h\rangle_e + \sum_{e\in\mathcal{E}^\Gamma} \langle \overline{\eta}_1 u_h, v_h\rangle_e \;; \end{aligned} \tag{21}$$

$$F_{DD}(v_h) = \sum_{T\in\mathcal{T}_h} (f, v_h)_T + \sum_{e\in\mathcal{E}^\Gamma} \langle g, (\zeta\boldsymbol{\nabla} v_h\cdot\boldsymbol{n})\rangle_e + \sum_{e\in\mathcal{E}^\Gamma} \langle \overline{\eta}_1 g, v_h\rangle_e - \sum_{e\in\mathcal{E}^{\Gamma-}} \langle (\boldsymbol{\beta}\cdot\boldsymbol{n}) g, v_h\rangle_e\,, \tag{22}$$

where $\overline{\eta}_1 = 4\zeta^*/s$ with $\zeta^* = \varepsilon + \xi_{h,max}^H$, $\xi_{h,max}^H = \max\left\{\xi_h^H(u_H|_{T_1}), \xi_h^H(u_H|_{T_2})\right\}$. Here, T_1 and T_2 are the macro elements that share an edge e.

We should remark that the operator (18) has much in common with the nonlinear term of the discontinuity capturing method proposed in Galeão and do Carmo (1988), although it has been developed by following a different approach, yielding a different artificial diffusion which only depends on the resolved scale degrees of freedom. Moreover, another remarkable difference is that the proposed two-scale framework yields a method with no stabilization coefficients and no extra linear stabilization term. It also differs from the NSDG method developed in Arruda et al. (2010) because the regularization provided by the model $\mathcal{M}_{DD}$ is applied on all scales. The consistency property comes directly from the fact that both the exact and the resolved scale solutions have no subgrid scales. The numerical experiments conducted in this paper show that the DD method presents optimal convergence rates and yields higher stable resolved scale solutions for general meshes and when the velocity field is not constant, outperforming many DG methods, with or without discontinuity capturing terms.

5 NUMERICAL RESULTS

In this section, two classical academic numerical experiments are conducted to illustrate the behavior of the proposed discontinuous formulations applied to advection-diffusion problems. The new formulations are compared here with the DG method, using the formulation developed in Houston et al. (2000). Since a continuous solution is desired for advection-diffusion equation, all the approximated solutions are represented in a continuous way, where the solution in each node of the mesh is the average solution of all corresponding degrees of freedom. In the following computational experiments the convergence of the nonlinear procedure is attained setting a tolerance equal to 10^{-3}.

5.1 Example 1: Parabolic and exponential layers

This example simulates a two-dimensional advection-dominated problem with $\varepsilon = 1 \times 10^{-4}$, $\boldsymbol{\beta} = (1, 0)$ and a constant source term $f = 1$ in $\Omega = (0, 1) \times (0, 1)$. The Dirichlet boundary conditions are homogeneous in Γ. The exact solution is an inclined plane having a 45^o slope, with parabolic layers at $y = 0$ and $y = 1$ and a exponential layer at $x = 1$.

Numerical results for a partition of the domain with 20 divisions in each side are presented in Figure 2. The comparison between the DCAU and DG solutions reveals that they yield similar behavior. Indeed, Figures 2(a) and 2(b) show that both methods give rise to spurious oscillations in the neighborhood of the parabolic layers. Such overshoots are damped by using the DD method, whose solution is depicted in Figure 2(c). The profiles at $y = 0.5$, $x = 0.5$, $y = x$ and $y = 1 - x$, presented in Figure 3, provide a better comparison among the methods. We may see that the DD solution is almost free of spurious modes at all sections.

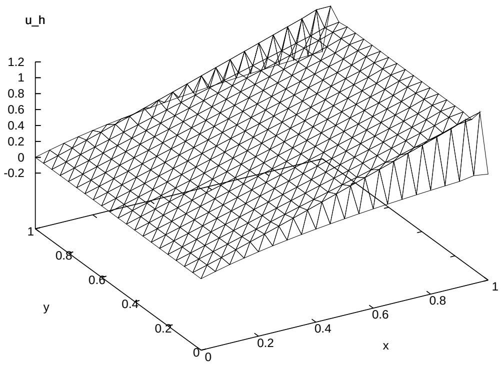
5.2 Example 2: Rotating pulse with internal and external layers

This example considers a transport problem with $\varepsilon = 10^{-6}$, $\boldsymbol{\beta} = (2y(1-x^2), -2x(1-y^2))$, $f = 0$ and $\Omega = (-1, 1) \times (0, 1)$. The inflow and outflow boundaries are $\{(x, 0); -1 \leq x \leq 0\}$ and $\{(x, 0); 0 \leq x \leq 1\}$, respectively. At the outflow, homogeneous natural boundary conditions are prescribed. There is a discontinuity at the inflow boundary, where Dirichlet boundary conditions are given by

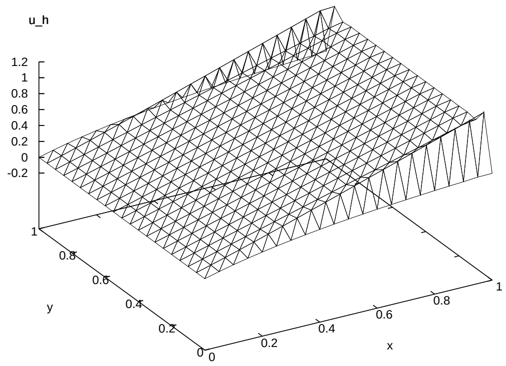
$$u(x,0) = \begin{cases} 0, & \text{if } -1 \leq x < -0.5; \\ 1, & \text{if } -0.5 \leq x \leq 0. \end{cases} \tag{23}$$

On the remaining boundary, homogeneous Dirichlet boundary conditions are set on the sides $x = -1$ and $y = 1$, and $u = 1$ is set at $x = 1$. The inflow discontinuity propagates across the domain and gives rise of a semi-circular internal layer. Moreover, an external layer appears at $x = 1$.

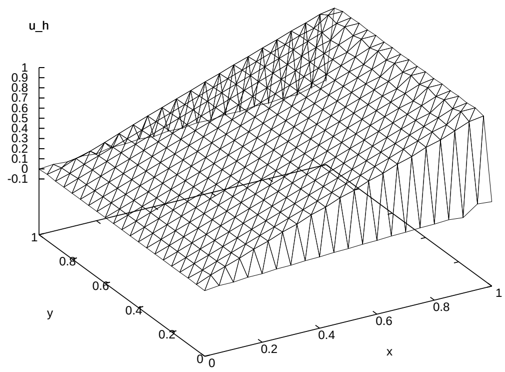
Figure 4 shows the approximate solutions obtained with the DG, DCAU and DD methods, using a partition of the domain with 48 and 24 divisions in x and y directions, respectively. The profiles at $y = 0.25$, $x = 0$, $y = x$ and $y = -x$ are presented in Figure 5. We may notice that again the DG and the DCAU methods present oscillations along the internal layer. As depicted in Figure 5, these oscillations are almost completely damped using the DD method, although



(a) DG Solution (u_h).

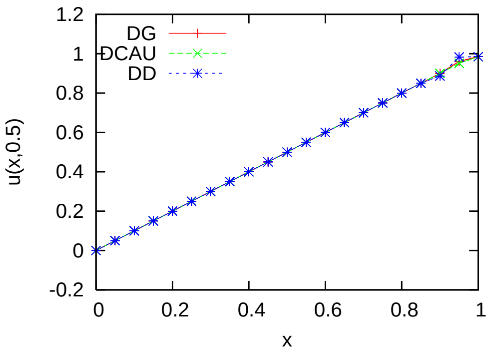


(b) DCAU Solution (u_h).

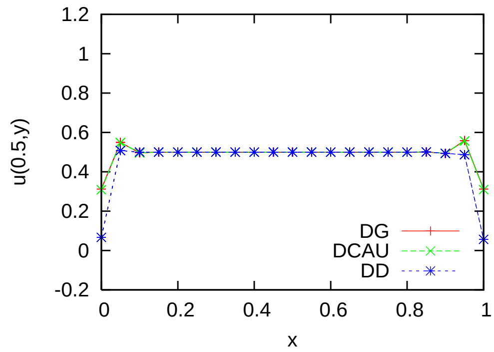


(c) DD Solution (u_h).

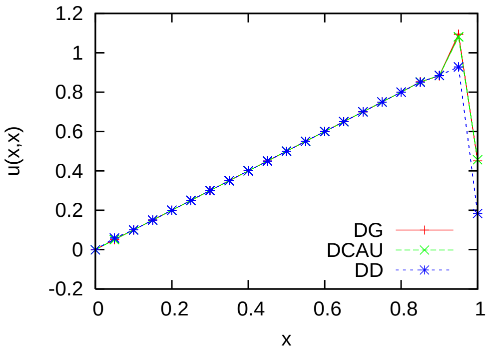
Figure 2: Example 1.



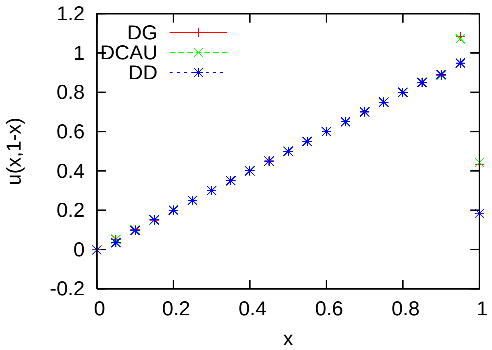
(a) Horizontal Section ($y = 0.5$).



(b) Vertical Section ($x = 0.5$)



(c) Diagonal Section ($y = x$).



(d) Diagonal Section ($y = 1 - x$).

Figure 3: Example 1 - Solution profiles.

we observe non-negligible oscillations at the inflow. This can be easily overcome by strongly imposing the inflow Dirichlet conditions.

5.3 Example 3: convergence rates

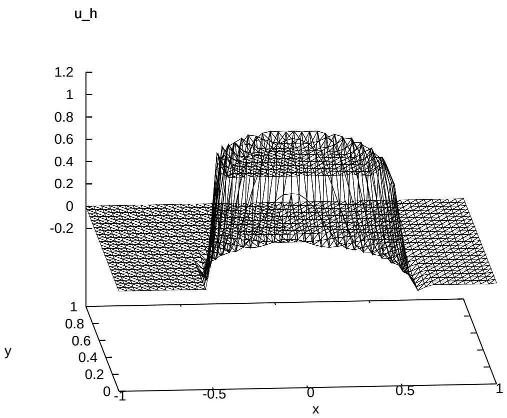
Finally, in this example we numerically evaluate the convergence properties of the proposed methodologies. It simulates and advection-diffusion problem with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-6}$ and $\boldsymbol{\beta} = (1, 0)$ in $\Omega = (0, 1) \times (0, 1)$. The source term and Dirichlet boundary conditions are chosen according to the smooth exact solution, given by

$$u(x, y) = sin(\pi x)sin(\pi y). \tag{24}$$

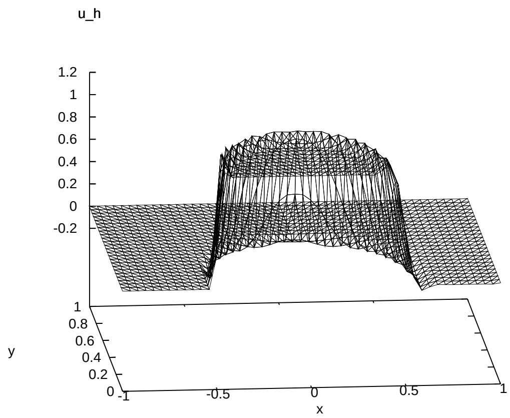
Figure 6 show the least-square linear approximation of the solution errors associated with the DG, DCAU and DD methods, measure in the $L^2(\Omega)$ and $H^1(\Omega)$ norms. Figures 6(a)-(b) show the results obtained with $\varepsilon = 10^{-3}$, while Figures 6(c)-(d) refer to $\varepsilon = 10^{-6}$. Optimal convergence rates are obtained for both methods, independently of the diffusivity coefficient value. As expected, the errors of the DG and DCAU methods are much alike and, since the DD method provides stronger regularization, it yields highest errors. We may also notice that the DD method presents slightly higher convergence rates than DG and DCAU methods, That difference is more pronounced for $\varepsilon = 10^{-3}$.

6 CONCLUSION

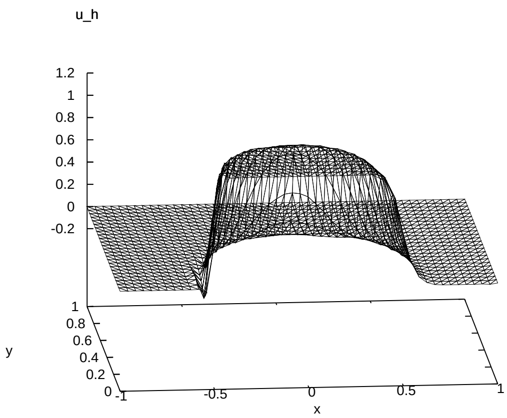
Two new discontinuous for the numerical solution of advection-diffusion-reaction problems are developed. The first one reformulates, using broken spaces, a shock capturing method



(a) DG Solution (u_h).

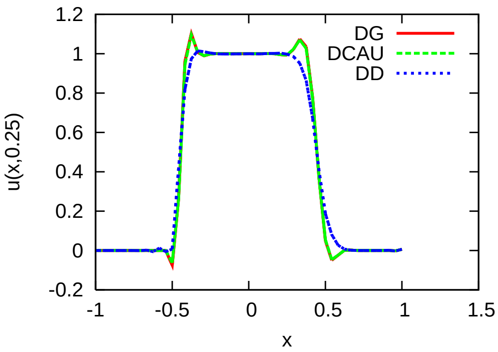


(b) DCAU Solution (u_h).

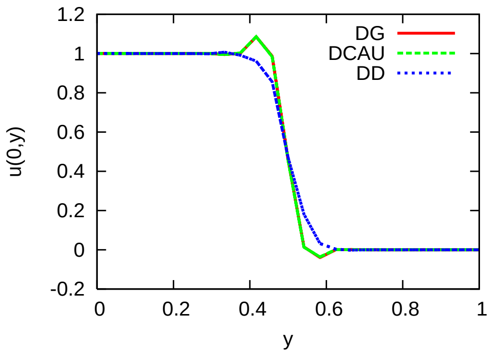


(c) DD Solution (u_h).

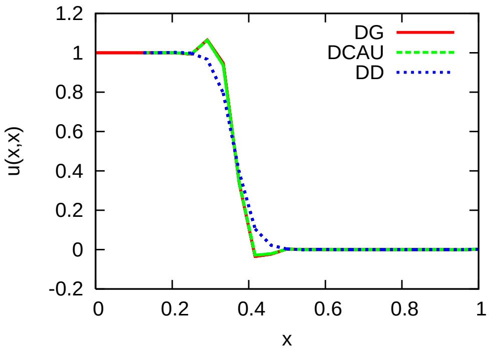
Figure 4: Example 2.



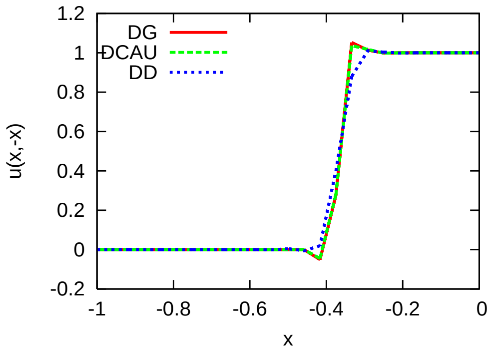
(a) Horizontal Section ($y = 0.25$).



(b) Vertical Section ($x = 0$)



(c) Diagonal Section ($y = x$).



(d) Diagonal Section ($y = -x$).

Figure 5: Example 2 - Solution profiles.

named Consistent Approximate Upwind Petrov-Galerkin (CAU) method. Although it has good stability properties when used in a conforming setting, its extension in the context of discontinuous Galerkin methods as performed here (the DCAU method) does not preserve similar property. Moreover, since it introduces a nonlinearity to the model solution, the disadvantage of this approach for solving linear transport problems is obvious. On the other hand, the nonlinear two-scale method proposed here improves by far the accuracy near sharp boundary layers whenever spurious oscillations have to be suppressed. The DD method provides stabilization by means of a local resolved scale residual-based stabilization and by a jump penalty that takes into account the effective flux through inter-element edges. The latter term guarantees optimal convergence rates for regular problems and the consistency property comes directly from the fact that the exact and the resolved scale solutions have no subgrid scales. A remarkable issue of this method is that its inherent adaptive ability to adjust the stabilization terms does not depend on any user defined stabilization parameter. Moreover, it reduces the number of degrees-of-freedom since discontinuities are allowed only on the resolved scale.

ACKNOWLEDGMENTS

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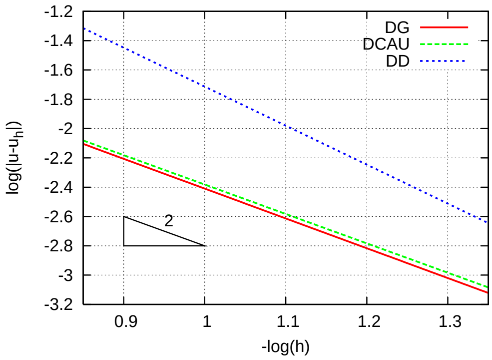
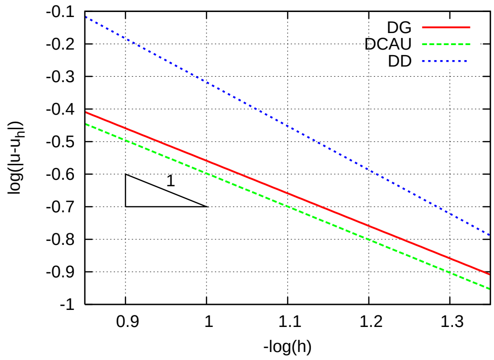
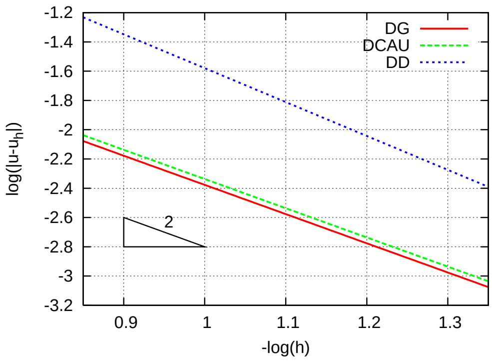


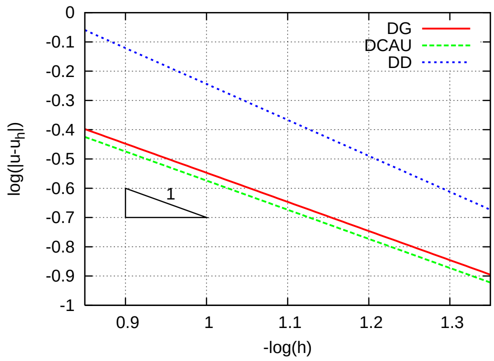
(a) $L^2(\Omega), \varepsilon = 10^{-3}$



(b) $H^1(\Omega), \varepsilon = 10^{-3}$



(c) $L^2(\Omega), \varepsilon = 10^{-6}$



(d) $H^1(\Omega), \varepsilon = 10^{-6}$

Figure 6: Convergence rates.

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