Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXX, págs. 225-242 (artículo completo)
Oscar Möller, Javier W. Signorelli, Mario A. Storti (Eds.)
Rosario, Argentina, 1-4 Noviembre 2011

PROPAGATION OF OCEAN WAVES OVER A SHELF WITH LINEAR TRANSITION

Eric G. Bautista, Emmanuel Arcos and Oscar E. Bautista

Instituto Politécnico Nacional, ESIME Azcapotzalco, México, D. F., 02250, México, http://www.sepi.esimeazc.ipn.mx/

Keywords: Reflection and transmission coefficients, ocean wave propagation, singular perturbation, dimensionless analysis.

Abstract: A singular perturbation analysis based on the WKB (Wentzel–Kramers–Brillouin) technique to study the hydrodynamic performance of periodic ocean waves that are incident on a shelf with linear transition is proposed. We derive a linear model to predict the propagation of the long ocean waves over the shelf. In this manner, the spatial distribution for the surface elevation of the ocean waves over the shelf as a function of three dimensionless parameters, namely, a small kinematical parameter and two geometrical parameters, is governed by a second-order linear ordinary differential equation. The kinematical parameter denotes the ratio of the potential head, due to gravity, to the kinetic head of the ocean waves along the longitudinal axis of the parabolic channel. Meanwhile the geometrical parameters represent a characteristic depth ratio of the shelf and the shelf slope. Using matching conditions, simple expressions for the reflection and transmission coefficients are obtained.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

1 INTRODUCTION

A wide variety of physical phenomena can be simulated by using shallow water mathematical models. An important class of problems of practical interest involves water flows with a free surface under the influence of gravity. This includes tides in oceans, the breaking of waves on shallow beaches, rolling waves in open channels, flood waves in rivers, surges and dam break wave modeling. Some of these relevant models can be found in Toro (2001) and Wu (1981).One manner to classify ocean waves is based on their relative depth. For the case of shallow water flows, waves for which $h/\lambda < 1/20$ or $kh < \pi/10$ are called long waves or shallow waves, as was proposed by Rahman (1995). In the above relationships, k denotes the wave number, λ is the ocean wave length and h represents a characteristic elevation of the ocean wave. An essential aspect of the propagation of ocean waves described by the shallow water approximation is that they are strongly influenced by geometrical and physical parameters. Among others, we can include sharp variations in the depth and width that are characteristic parameters of the specific geometry of the open channel and other factors such as friction and wind velocity effects that can also affect the motion of ocean waves, as was reported by Li and Jeng (2006). It is well known that one of the principal obstacles in obtaining adequate analytical solutions lies in the complicated geometry of natural estuaries; however, for some simplified geometries, it is possible to develop approximate analytical solutions. On the other hand, the modern use of digital computers can drastically reduce the numerical difficulties associated with complex geometries by enabling variations in depth and breadth to be incorporated in the calculations. This in turn increases the accuracy in the numerical solutions. Today, computational methods offer a sophisticated tool for studying the dynamic propagation of shallow water waves. However, for some relevant limits, the analytical solutions provide knowledge about the phenomenon of interest and can be very useful for simplifying numerical models. Therefore, the use of numerical methods does not detract the fundamental role of approximate analytical solutions. In this direction, Aubrey and Speer (1985) theoretically examined the spreading and distortion of long waves into shallow estuarine structures. Their observations were based on the sea surface elevation and horizontal currents over periods ranging from three days to one year, documenting the non-linear interactions associated with the offshore equilibrium. Some authors have treated the modeling of ocean waves indifferent ways. For instance, the mathematical analysis of long ocean waves characterizing discontinuous density gradients was reported by Ouahsine and Bois (2001). They argued that the mathematical structure of the pressure perturbation associated with the leading motion can be reduced to a Sturm–Liouville problem with well-defined eigenvalues. Mead (2004) proposed a numerical solution to the shallow water equation by using an analytical scheme with Lagrangian coordinates and plotting the particles trajectories to avoid the use of complicated generation meshes. In some cases, shock waves are present in long waves and these can be treated by using the well-known Riemann solver for a hyperbolic system in the context of two-dimensional conservation laws. Contributions related to the above situations can be seen elsewhere see Baghlani et al. (2008), Guinot (2005) and George (2008). In parallel, Läuter et al. (2005) developed a new class of unsteady analytical solutions to a spherical shallow water equation using the transformation method. Camassa et al. (2006), introduced an asymptotic higher-order model for the wave dynamics of a shallow water problem by combining analytical and numerical analyses. Bautista et al. (2011), proposed a singular perturbation analysis to study the hydrodynamic performance of periodic ocean waves that are incident on an open parabolic

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

channel of constant depth. O'Hare and Davies (1992), presented a model for the monochromatic propagation on a smoothly varying bed profile divided into a series of shelves separated by abrupt steps. Wang et al. (2002), proposed a three-point method for estimating wave reflection to account for monochromatic oblique incident waves propagating over a sloping beach. The submerged obstacles with different shapes, solids and porous solids have been studied extensively in order to determine the reflection and transmission of ocean waves, and representative works can be found elsewhere see Losada et al. (1989), Pengzhi (2004), Pengzhi and Liu (2005), Wang et al. (2008).

The ocean wave reflection is a phenomenon that has to be taken into account in the ocean wave mechanic, due to the importance in the design and construction of coastal structures. In the past, different researchers conducted fundamental contributions on this topic related to the propagation of ocean waves in complex geometries with the aid of experimental and theoretical models to obtain the reflection and transmission coefficients.

In this direction, there are a number of analytical studies for the transformation of ocean waves, one important case that had been studied widely is the ocean waves propagating over an infinite step, Lamb (1932), presented the long-wave solution to the reflection or transmission of waves for an infinite step by using matching conditions for surface and normal mass flux at the boundary. In the specialized literature, approximated analytic solutions for the long wave equations with different propagation conditions based on Bessel functions can be found elsewhere, see Goring (1979), Bender and Dean (2003), Dean and Dalrymple (2004), Hsien-Kuo and Jin-Cheng (2007), Tae-Hwa and Yong-Sik (2009).

In this work, we develop a new theoretical model for studying the spreading of ocean waves over a shelf with linear transition. We consider the lineal shallow water flow approximation by taking into account the influence of the bottom variation on wave propagation, the model is based under a potential flow theory. Special attention is placed on the effects of the perturbation for a small amplitude ocean wave propagating over the shelf. The governing equations for the ocean wave propagation are presented in dimensionless form as a function of dimensionless parameters. For this purpose, we apply the well-known WKB (Wentzel–Kramers–Brillouin) perturbation technique, see Bender and Orzag (1978). In order to obtain the reflection and transmission coefficients and considering that our physical model is subdivided into three different regions, we anticipate that the continuity of kinematic and dynamic conditions will prevail, the reflection and transmission coefficient are function of the relative depth, shelf slope and the small kinematic parameter. For instance, the surface elevation and mass flow between each pair of regions must be the same, as was originally proposed by Mei et al. (2005). The new analytical solution is a versatile tool that can be extended to analyze engineering structures as breakwater with trapezoidal shapes or any continuous bottom shapes.

2 FORMULATION

The physical model under study is shown in Fig. 1. In the Cartesian coordinate system the x axis points in a direction of the incident waves, with the z axis pointing outwards in a direction normal to the mean sea-water level. In addition, the physical model is divided into three different regions: R1, R2 and R3, with prescribed depths defined respectively by the following relationships:

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

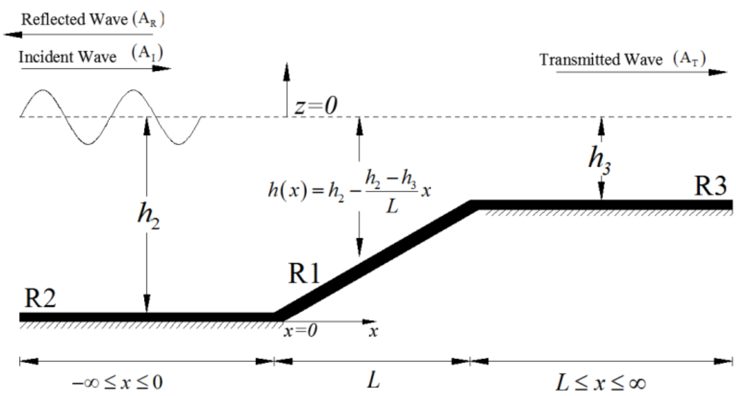

Fig. 1. Transversal section of the physical model

$$h(x)=\begin{bmatrix} \text{R2: } h_2 & \text{For } x\le 0 \\ \text{R1: } h_2-\dfrac{h_2-h_3}{L}x & \text{For } 0\le x\le L \\ \text{R3: } h_3 & \text{For } x\ge L \end{bmatrix} \tag{1}$$

where L is the horizontal projection length of region R1, for regions R2 and R3 the x axis extends from $-\infty \le x \le 0$ and $L \le x \le \infty$, respectively. At the bottom, we assume that the walls are impermeable which means no mass transfer. On the other hand, the boundary conditions on the right and left sides of the system are completely open to the passage of the sum of incident and reflected waves on the left and only a transmitted wave on the right. As stated previously, in the present study, we assume that the following two shallow water equations (SWE), under the theory of potential flow and assuming small amplitudes, can be written as follows:

$$\frac{\partial U(x,t)}{\partial t} = -g\frac{\partial \eta(x,t)}{\partial x} \tag{2}$$

$$\frac{\partial \eta(x,t)}{\partial t} = -\frac{\partial\left[U(x,t)h(x)\right]}{\partial x} \tag{3}$$

The first equation represents the conservation of momentum, and the second one represents the conservation of mass. In the above equations, $U(x,t)$ is the mean velocity, $\eta(x,t)$ is the level of the water surface measured from the sea level at rest, t is the physical time, g is gravitational acceleration and $h(x)$ is the depth variation. The above set of equations can be written for each region by using appropriate dimensionless variables, and together with the boundary conditions, define the basic characteristics of the present model.

Combining Eqs. (2) and (3), the motion equation for region R1 is given by the following:

$$\frac{\partial^2 \eta}{\partial t^2} = g\left[h(x)\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \eta}{\partial x}\frac{\partial h(x)}{\partial x}\right] \tag{4}$$

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

Taking into account the relationships, Eq. (1), for the interval $0 \le x \le L$, Eq. (4) become

$$\frac{\partial^2 \eta}{\partial t^2} = g\left\{\left(h_2 - \frac{h_2 - h_3}{L}x\right)\frac{\partial^2 \eta}{\partial x^2} - \left(\frac{h_2 - h_3}{L}\right)\frac{\partial \eta}{\partial x}\right\} \tag{5}$$

Eq. (5) describes the unsteady oscillations of the water surface level. The above equation can be simplified in order to reduced to an ordinary differential equation by considering that the surface level $\eta(x,t)$ has a periodic response and modulated only by the variable amplitude as follows:

$$\eta(x,t) = \bar{\eta}(x)\cos(\omega t) \tag{6}$$

where $\bar{\eta}(x)$ is the wave amplitude in any cross section along the coordinate x, $\omega = 2\pi / T$ is the ocean wave angular frequency and T is the period.

Substituting Eq. (6) into Eq.(5), we obtain the following

$$g\left[\left(h_2 - \frac{h_2 - h_3}{L}x\right)\frac{d^2\eta}{dx^2} - \left(\frac{h_2 - h_3}{L}\right)\frac{d\eta}{dx}\right] + \omega^2\bar{\eta} = 0 \tag{7}$$

In the following section, we present a dimensionless version of the equations to clarify the influence of the dimensionless parameters.

We can write Eq. (7) in dimensionless form by introducing the following variables:

$$\bar{\chi} = \frac{x}{\sqrt{L^2 + (h_2 - h_3)^2}}, \qquad \bar{\delta} = \frac{\bar{\eta}(x)}{h_2}$$

substituting the previous dimensionless variables into Eq. (7) we obtain:

$$\kappa\frac{d^2\bar{\delta}}{d\bar{\chi}^2} - \frac{\kappa\alpha}{(1-\alpha\bar{\chi})}\frac{d\bar{\delta}}{d\bar{\chi}} + \frac{1}{(1-\alpha\bar{\chi})}\bar{\delta} = 0 \tag{8}$$

where the dimensionless parameters in the above equation are defined as

$$\kappa = \frac{g\,h_2}{\omega^2\left[L^2 + (h_2 - h_3)^2\right]}, \quad \alpha = \sqrt{1+m^2}\,(1-\varepsilon), \quad m = \frac{h_2 - h_3}{L} \quad \text{and} \quad \varepsilon = \frac{h_3}{h_2}$$

Eq. (8) is a singular boundary value problem, which describe the oscillation of the water surface level as a function of the dimensionless variable $\bar{\chi}$.

3 WKB THEORY: ASYMPTOTIC LIMIT FOR $\kappa \ll 1$

In this section, the asymptotic limit for $\kappa \ll 1$ using the WKB perturbation technique is analyzed in order to obtain an approximated analytical solution of Eq. (8), which is an ordinary linear second order differential equation with the highest-order derivative multiplied by the small parameter κ. It should be noted that the physical meaning of the limit κ tends to zero is basically interpreted as a shallow water wave propagation for which the potential head, gh_2 is dominated by the kinetic head, $\omega^2\left[L^2 + (h_2 - h_3)^2\right]$, associated to the frequency ω of the wave. This case physically corresponds to a well-established propagation of the water waves. In addition, we must emphasize the fundamental importance for selecting the dimensionless form of the governing equations derived in the previous section: WKB theory

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

is a powerful tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter. However, the useof the WKB technique is not restricted if the differential equation is written in dimensional or dimensionless form. Therefore, the reason for applying the WKB method to the dimensionless form of governing equations is only based on the fact that the corresponding dimensionless form of the equations is usual in most theoretical analyses of fluid mechanics. In our case, the perturbation parameter in Eq. (8) is given by the kinematical parameter, κ. Therefore, the mathematical structure of this dispersive problem clearly defines a well-known singular behavior which can be analyzed by the proposed perturbative technique.

In order to applied the WKB perturbation technique, using the Liouville transformation the Eq.(8) is transformed in a canonical form , as follows, see Milson (1997).

$$\tilde{\phi}''(\bar{\chi})+Q(\bar{\chi})\tilde{\phi}(\bar{\chi})=0 \tag{9}$$

where $\tilde{\phi}(\bar{\chi})$ correspond to a change of variable of the form

$$\tilde{\phi}(\bar{\chi})=\bar{\delta}(\bar{\chi})e^{\int_0^{\bar{\chi}}\frac{b(\bar{\chi})}{2a(\bar{\chi})}d\bar{\chi}} \tag{10}$$

The variable coefficient $Q(\bar{\chi})$ in Eq. (9) is calculated as follows

$$Q(\bar{\chi})=\frac{\left[4a_{\bar{\chi}}c_{\bar{\chi}}-2a_{\bar{\chi}}b'_{\bar{\chi}}+2b_{\bar{\chi}}a'_{\bar{\chi}}-b_{\bar{\chi}}^2\right]}{4a_{\bar{\chi}}^2} \tag{11}$$

In Eqs. (10) and (11), the variable coefficients $a_{\bar{\chi}}$, $b_{\bar{\chi}}$ and $c_{\bar{\chi}}$, correspond to the coefficients of Eq. (8) as follows:

$$a=\kappa,\quad a'=0,\quad b=-\frac{\kappa\alpha}{1-\alpha\bar{\chi}},\quad b'=-\frac{\kappa\alpha^2}{(1-\alpha\bar{\chi})^2}\quad \text{and}\quad c=\frac{1}{1-\alpha\bar{\chi}}$$

Taking into account the above relationship in Eq. (11) and Eq. (10), we obtain the following

$$Q(\bar{\chi})=\frac{1}{\kappa(1-\alpha\bar{\chi})}+\frac{\alpha^2}{4(1-\alpha\bar{\chi})^2} \tag{12}$$

and

$$\tilde{\phi}(\bar{\chi})=\bar{\delta}(\bar{\chi})(1-\alpha\bar{\chi})^{1/2}E \tag{13}$$

where E is a constant of integration.

Substituting Eq.(12) into Eq. (9), and multiplying the result with κ we obtain the canonical form of Eq. (8) as follows:

$$\kappa\tilde{\phi}''(\bar{\chi})+\left[\frac{1}{(1-\alpha\bar{\chi})}+\frac{\kappa\alpha^2}{4(1-\alpha\bar{\chi})^2}\right]\tilde{\phi}(\bar{\chi})=0 \tag{14}$$

An approximated WKB analytical solution for Eq. (14) can be proposed as a potential series given by the following:

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

$$\tilde{\phi}(\bar{\chi}) \sim \exp\left[\frac{1}{\varsigma}\sum_{n=0}^{\infty}\varsigma^n S_n(\bar{\chi})\right], \quad \varsigma \to 0; \tag{15}$$

with the first and second derivatives given by the following:

$$\begin{aligned} \tilde{\phi}'(\bar{\chi}) &\sim \left[\frac{1}{\varsigma}\sum_{n=0}^{\infty}\varsigma^n S'_n(\bar{\chi})\right]\exp\left[\frac{1}{\varsigma}\sum_{n=0}^{\infty}\varsigma^n S_n(\bar{\chi})\right], \qquad \varsigma \to 0 \\ \tilde{\phi}''(\bar{\chi}) &\sim \left[\frac{1}{\varsigma^2}\left(\sum_{n=0}^{\infty}\varsigma^n S'_n(\bar{\chi})\right)^2 + \frac{1}{\varsigma}\sum_{n=0}^{\infty}\varsigma^n S''_n(\bar{\chi})\right]\exp\left[\frac{1}{\varsigma}\sum_{n=0}^{\infty}\varsigma^n S_n(\bar{\chi})\right], \quad \varsigma \to 0; \end{aligned} \tag{16}$$

where S_n, the phase function, is an unknown function of the dimensionless kinematical parameter κ and must be determined for each value of n, as a part of the mathematical procedure.

Obviously, the phase S_n is assumed to be a non-constant function and slowly varying in a break down region. Taking into account the solution given by Eq. (15) and the relationships in Eq.(16), Eq.(14) leads to the following expression:

$$\frac{\kappa}{\varsigma^2}\left(S_0'\right)^2 + \frac{\kappa}{\varsigma}\left(2S_0'S_1'\right) + \kappa\left(S_1'\right)^2 + \frac{\kappa}{\varsigma}S_0'' + \kappa\left(S_1''\right) + \left[\frac{1}{(1-\alpha\bar{\chi})} + \frac{\kappa\alpha^2}{4(1-\alpha\bar{\chi})}\right] = 0 \tag{17}$$

and by a simple dominant balance, we obtain the following:

$$\varsigma = \kappa^{1/2} \tag{18}$$

in this manner, Eq. (17) can now be written as

$$\kappa^0\left[S_0'^2 + \frac{1}{(1-\alpha\bar{\chi})}\right] + \kappa^{1/2}\left[2S_0'S_1' + S_0''\right] + \kappa^1\left[S_1'^2 + S_1'' + \frac{\kappa\alpha^2}{4(1-\alpha\bar{\chi})}\right] = 0 \tag{19}$$

Therefore, from Eq. (19), we can collect terms of order κ^0 and $\kappa^{1/2}$, and the following equations up to terms of order $\kappa^{1/2}$, are obtained as follows:

$$\kappa^0 : S_0'^2 + \frac{1}{(1-\alpha\bar{\chi})} = 0, \tag{20}$$

and

$$\kappa^{1/2} : 2S_0'S_1' + S_0'' = 0 \tag{21}$$

Eq. (20) yields the solution for S_0' as follows:

$$S_0' = \pm\frac{i}{(1-\alpha\bar{\chi})} \tag{22}$$

where $i = (-1)^{1/2}$, and thus, the solutions for S_0 are given by the following relationships:

$$S_0 = -\frac{2i}{\alpha}(1-\alpha\bar{\chi})^{1/2} + C, \tag{23}$$

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

and

$$S_0 = \frac{2i}{\alpha}\left(1-\alpha\bar{\chi}\right)^{1/2} + C, \tag{24}$$

The constant C can be determined with appropriated boundary conditions.
In a similar way, from Eq. (21), $S_1^{'}$ is obtained as follows:

$$S_1^{'} = -\frac{\alpha}{4\left(1-\alpha\bar{\chi}\right)^{1/2}} \tag{25}$$

and therefore, S_1 is given by

$$S_1 = \frac{1}{4} Ln\left(1-\alpha\bar{\chi}\right) + D \tag{26}$$

In Eq. (26), D is another integration constant that is also determined by the boundary conditions. The procedures for estimating both constants are given below. Substituting Eqs.(24) and (26) into Eq. (15), the WKB solution to the modified differential equation, Eq. (14), is given by the following:

$$\tilde{\phi}\left(\bar{\chi}\right) = \left(1-\alpha\bar{\chi}\right)^{1/4}\left(C\, e^{\frac{2\left(1-\alpha\bar{\chi}\right)^{1/2}}{\kappa^{1/2}\alpha} i} + D\, e^{-\frac{2\left(1-\alpha\bar{\chi}\right)^{1/2}}{\kappa^{1/2}\alpha} i}\right), \quad \kappa \to 0. \tag{27}$$

Finally, substituting Eq. (27) into Eq. (13), we obtain the solution to the original differential equation Eq. (8)

$$\bar{\delta}\left(\bar{\chi}\right) = \frac{1}{\left(1-\alpha\bar{\chi}\right)^{1/4}}\left(C\, e^{\frac{2\left(1-\alpha\bar{\chi}\right)^{1/2}}{\kappa^{1/2}\alpha} i} + D\, e^{-\frac{2\left(1-\alpha\bar{\chi}\right)^{1/2}}{\kappa^{1/2}\alpha} i}\right), \quad \kappa \to 0. \tag{28}$$

The determination of constants *C* and *D* will deduced in the next sections.

For the regions of uniform depth, R2 and R3, Eq. (4)simplifies to

$$\frac{\partial^2 \eta}{\partial t^2} = g h_{2,3} \frac{\partial^2 \eta}{\partial x^2} \tag{29}$$

where $h_{2,3}$ is the constant depth of region R2 and R3.

Eq. (29) is reduced to a boundary value problem, considering that the surface level has a periodic response and is modulated only by the variable amplitude; thus, we propose asolution given by

$$\eta = \tilde{\eta}_{2,3} \cos\left(\omega\tau\right) \tag{30}$$

substituting Eq. (30) into Eq. (29), we obtain

$$\frac{d^2 \tilde{\eta}_{2,3}}{dx} + \frac{\omega^2}{g h_{2,3}} \tilde{\eta}_{2,3} = 0 \tag{31}$$

where $\tilde{\eta}_{2,3}$ is the ocean wave amplitude in any transversal section of the region R2 and R3, respectively.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

The above equation has a direct solution. A general solution of Eq. (31) can be expressed as follows:

$$\tilde{\eta}_2(x) = A_I e^{\frac{\omega}{(gh_2)^{1/2}} i} + A_R e^{-\frac{\omega}{(gh_2)^{1/2}} i} \tag{32}$$

and

$$\tilde{\eta}_3(x) = A_T e^{\frac{\omega}{(gh_3)^{1/2}} i} \tag{33}$$

where A_I is the known physical incident ocean wave amplitude , A_R and A_T are the unknown physicals reflected an transmitted ocean waves amplitudes, that have to be determined.

Introducing the next dimensionless variables for region R2 and R3

$$\tilde{\chi} = \frac{x}{(gh_2)^{1/2}/\omega}, \qquad \tilde{\delta} = \frac{\tilde{\eta}_2}{h_2} \quad \text{and} \quad \hat{\chi} = \frac{x}{(gh_3)^{1/2}/\omega}, \qquad \hat{\delta} = \frac{\tilde{\eta}_3}{h_3}$$

the corresponding Eq. (32) and (33) are written in dimensionless variables as follows:

$$\tilde{\delta}(\tilde{\chi}) = \beta_I e^{\tilde{\chi} i} + \beta_R e^{-\tilde{\chi} i} \tag{34}$$

and

$$\hat{\delta}(\hat{\chi}) = \beta_T e^{\hat{\chi} i} \tag{35}$$

where $\beta_I = A_I / h_2$, $\beta_R = A_R / h_2$ and $\beta_T = A_T / h_3$. The complex constants β_R and β_T have to be determined with appropriated boundary conditions.

The procedures for estimating the constants *C*, *D* , β_R and β_T given by the equations (28), (34) and (35) are given below.

4 REFLECTION AND TRANSMISSION COEFFICIENTS

Matching conditions must be applied to conserve the surface elevation and flow fluxes across in commune regions. These are given by the following relationships:

For regions R2 and R1

$$\tilde{\delta}\Big|_{\tilde{\chi}=0} = \bar{\delta}\Big|_{\bar{\chi}=0} \quad \text{and} \quad \frac{d\tilde{\delta}}{d\tilde{\chi}}\bigg|_{\tilde{\chi}=0} = \kappa^{1/2} \frac{d\bar{\delta}}{d\bar{\chi}}\bigg|_{\bar{\chi}=0} \tag{36}$$

and for regions R1 and R3

$$\bar{\delta}\Big|_{\bar{\chi}=\frac{1}{\sqrt{1+m^2}}} = \varepsilon\,\hat{\delta}\Big|_{\hat{\chi}=\frac{\omega L}{(gh_3)^{1/2}}} \quad \text{and} \quad \kappa^{1/2} \frac{d\bar{\delta}}{d\bar{\chi}}\bigg|_{\bar{\chi}=\frac{1}{\sqrt{1+m^2}}} = \varepsilon^{1/2} \frac{d\hat{\delta}}{d\hat{\chi}}\bigg|_{\hat{\chi}=\frac{\omega L}{(gh_3)^{1/2}}} \tag{37}$$

In the matrix form, the above equations are written as

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

$$\begin{bmatrix} -1 & e^{\frac{2}{\kappa^{1/2}\alpha}i} & e^{-\frac{2}{\kappa^{1/2}\alpha}i} & 0 \\ 1 & -e^{\frac{2}{\kappa^{1/2}\alpha}i}\left(1+\frac{\kappa^{1/2}\alpha}{4}i\right) & e^{-\frac{2}{\kappa^{1/2}\alpha}i}\left(1-\frac{\kappa^{1/2}\alpha}{4}i\right) & 0 \\ 0 & e^{\frac{2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i} & e^{-\frac{2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i} & -\varepsilon^{5/4}e^{\frac{\omega L}{(gh_3)^{1/2}}i} \\ 0 & -e^{\frac{2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i}\left(1+\frac{\kappa^{1/2}\alpha}{4\varepsilon^{1/2}}i\right) & e^{-\frac{2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i}\left(1-\frac{\kappa^{1/2}\alpha}{4\varepsilon^{1/2}}i\right) & -\varepsilon^{5/4}e^{\frac{\omega L}{(gh_3)^{1/2}}i} \end{bmatrix} \begin{bmatrix} \beta_R \\ C \\ D \\ \beta_T \end{bmatrix} = \begin{bmatrix} \beta_I \\ \beta_I \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

the solution of the system equations (38), yields the following relationships:

$$\frac{\beta_R}{\beta_I} = \frac{\begin{matrix} -\kappa^{1/2}\alpha\left(\frac{1}{\varepsilon^{1/2}}+1\right)\sin(\varphi)+ \\ +i\left[\kappa^{1/2}\alpha\left(\frac{1}{\varepsilon^{1/2}}-1\right)\cos(\varphi)-\frac{\kappa\alpha^2}{4\varepsilon^{1/2}}\sin(\varphi)\right] \end{matrix}}{\begin{matrix} -8\cos(\varphi)+\kappa^{1/2}\alpha\left(1-\frac{1}{\varepsilon^{1/2}}\right)\sin(\varphi)+ \\ +i\left[\kappa^{1/2}\alpha\left(1-\frac{1}{\varepsilon^{1/2}}\right)\cos(\varphi)+\left(\frac{\kappa\alpha^2}{4\varepsilon^{1/2}}+8\right)\sin(\varphi)\right] \end{matrix}} \tag{39}$$

and

$$\frac{\beta_T}{\beta_I} = \frac{-1}{\left\{\varepsilon^{5/4}\right\}\left\{\begin{matrix} -\cos(\varphi)+\frac{\kappa^{1/2}\alpha}{8}\left(1-\frac{1}{\varepsilon^{1/2}}\right)\sin(\varphi)+ \\ +i\left[\frac{\kappa^{1/2}\alpha}{8}\left(1-\frac{1}{\varepsilon^{1/2}}\right)\cos(\varphi)+\left(\frac{\kappa\alpha^2}{32\varepsilon^{1/2}}+1\right)\sin(\varphi)\right] \end{matrix}\right\}} e^{-\theta i} \tag{40}$$

where $\varphi = 2\left(1-\varepsilon^{1/2}\right)/\left(\kappa^{1/2}\alpha\right)$ and $\theta = \omega L/\left(gh_3\right)^{1/2}$

The constants C and D are given as follows:

$$C = \frac{\beta_I\left(\varepsilon^{-1/2}\kappa^{1/2}\alpha\, i\right)e^{\frac{-2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i}}{\frac{\kappa\alpha^2}{8\varepsilon^{1/2}}e^{\varphi i}-\left[8+\frac{\kappa\alpha^2}{8\varepsilon^{1/2}}+\kappa^{1/2}\alpha\left(\frac{1}{\varepsilon^{1/2}}-1\right)i\right]e^{-\varphi i}}, \tag{41}$$

and

$$D = \frac{-4\beta_I\left(1+\frac{\kappa^{1/2}\alpha}{8\varepsilon^{1/2}}i\right)e^{\frac{2\varepsilon^{1/2}}{\kappa^{1/2}\alpha}i}}{\frac{\kappa\alpha^2}{16\varepsilon^{1/2}}e^{\varphi i}-\left[4+\frac{\kappa\alpha^2}{16\varepsilon^{1/2}}+\frac{\kappa^{1/2}\alpha}{2}\left(\frac{1}{\varepsilon^{1/2}}-1\right)i\right]e^{-\varphi i}} \tag{42}$$

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

In order to calculate the reflection and transmission coefficient of long ocean waves spreading over a shelf in the limit of $\kappa \ll 1$, in this work we considered the wave energy times the group velocity concept, that is

$$C_2 E_{\text{Wave Reflected}} + C_3 E_{\text{Wave Transmitted}} = C_2 E_{\text{Wave Incident}}; \tag{43}$$

where $E_{\text{Wave Incident}} = \rho g A_I^2/2$, $E_{\text{Wave Reflected}} = \rho g A_R^2/2$ and $E_{\text{Wave Transmitted}} = \rho g A_T^2/2$ are the energy of the incident, reflected and transmitted ocean waves, respectively. For linear nondispersives long waves the group velocity is given by $C_2 = \sqrt{gh_2}$ and $C_3 = \sqrt{gh_3}$, thus Eq. (43) become

$$\frac{\rho g A_R{}^2}{2}\sqrt{gh_2} + \frac{\rho g A_T{}^2}{2}\sqrt{gh_3} = \frac{\rho g A_I{}^2}{2}\sqrt{gh_2} \tag{44}$$

dividing Eq. (44) by $\left(\rho g A_I{}^2/2\right)\sqrt{gh_2}$, we obtain

$$\left(\frac{A_R}{A_I}\right)^2 + \frac{\sqrt{h_3}}{\sqrt{h_2}}\left(\frac{A_T}{A_I}\right)^2 = 1 \tag{45}$$

The first and the second term on the left side in the above equation, represent the percentage of the energy that is reflected and transmitted, these values are always positive quantities, therefore the Eq. (45), can be expressed in terms of the absolute value as follows

$$\left|\frac{A_R}{A_I}\right|^2 + \frac{\sqrt{h_3}}{\sqrt{h_2}}\left|\frac{A_T}{A_I}\right|^2 = 1 \tag{46}$$

Eq. (46) is written in a simplified form as

$$C_R^2 + C_T^2 = 1 \tag{47}$$

where C_R and C_T are the reflection and transmission coefficient respectively under the concept of energy conservation and are given by the next expressions:

$$C_R = \left|\frac{A_R}{A_I}\right| \tag{48}$$

and

$$C_T = \left|\frac{A_T}{A_I}\right|\left(\frac{h_3}{h_2}\right)^{1/4} \tag{49}$$

Now, substituting the values of $\beta_I = A_I / h_2$, $\beta_R = A_R / h_2$ and $\beta_T = A_T / h_3$ into Eqs. (48) and (49) the reflection and transmission coefficient are rewritten in the next form

$$C_R = \left|\frac{\beta_R}{\beta_I}\right| \tag{50}$$

and

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

$$C_T = \left|\frac{\beta_T}{\beta_I}\right| \varepsilon^{5/4} \tag{51}$$

Now, substituting the Eqs. (39) and (40) into the above equations and taking into account that the absolute value of a complex number is given by its modulus, Eqs. (50) and (51) can be written as follows

$$C_R = \frac{\sqrt{\left\{\kappa^{1/2}\alpha\left(\frac{1}{\varepsilon^{1/2}}+1\right)\sin(\varphi)\right\}^2 + \left\{\kappa^{1/2}\alpha\left(\frac{1}{\varepsilon^{1/2}}-1\right)\cos(\varphi) - \frac{\kappa\alpha^2}{4\varepsilon^{1/2}}\sin(\varphi)\right\}^2}}{\sqrt{\left\{-8\cos(\varphi)+\kappa^{1/2}\alpha\left(1-\frac{1}{\varepsilon^{1/2}}\right)\sin(\varphi)\right\}^2 + \left\{\kappa^{1/2}\alpha\left(1-\frac{1}{\varepsilon^{1/2}}\right)\cos(\varphi) + \left(\frac{\kappa\alpha^2}{4\varepsilon^{1/2}}+8\right)\sin(\varphi)\right\}^2}} \tag{52}$$

and

$$C_T = \frac{1}{\sqrt{\left\{-\cos(\varphi)+\frac{\kappa^{1/2}\alpha}{8}\left(1-\frac{1}{\varepsilon^{1/2}}\right)\sin(\varphi)\right\}^2 + \left\{\frac{\kappa^{1/2}\alpha}{8}\left(1-\frac{1}{\varepsilon^{1/2}}\right)\cos(\varphi) + \left(\frac{\kappa\alpha^2}{32\varepsilon^{1/2}}+1\right)\sin(\varphi)\right\}^2}} \tag{53}$$

5 RESULTS AND DISCUSSION

In this work, we have analytically solved a simplified mathematical model for the linear shallow water ocean wave equations. Although in the past similar problems have widely been studied by using conventional techniques for other types of different geometries to those used in this work, see Bokhove (2005) and Prokof (2002), we have introduced a dimensionless motion equation for predicting the hydrodynamic behavior of the surface elevation of the ocean wave. The reason for choosing dimensionless quantities lies in the fact that they are pure parameters, and this in turn allows us to reduce the number of required physical variables and parameters.

In this study, there are two physical limitations to the applications derived from the present theory that should be noted. First, the ocean wave propagation is parallel to the longitudinal axis x; therefore, the mathematical solution does not provide oblique incident waves. Second, the mathematical model is based on a linearized theory of long waves, i.e., $\eta / h(x) \ll 1$ and $h(x)/\lambda \ll 1$.

For the energy conservation concept, the magnitude of the reflection coefficient is in the interval $0 \le C_R \le 1$ and transmission coefficient is in the interval $0 \le C_T \le 1$, when the ratio $\varepsilon \to 1$ and the shelf slope $m \to 0$ (Horizontal bottom) the Eqs. (52) and (53) take the values $C_R = 0$, there is not reflection and $C_T = 1$, perfect transmission as expected. Under these conditions, the governing equation of the propagation of ocean waves given by the Eq.(8) takes the form:

$$\kappa \frac{d^2\bar{\delta}}{d\bar{\chi}} + \bar{\delta} = 0 \tag{54}$$

The above equations describe a boundary value problem for the propagation of long waves over a region with flat bottom and is similar to those obtained for regions R2 and R3.

For values of $m \to \infty$ and depth ratio $\varepsilon \ll 1$, the shelf geometry degenerate into an infinite step, this case has been widely studied using different techniques. The wave propagation over

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

a step is a singular problem that needs to be studied as a particular case, in this work for values $\varepsilon \ll 1$ and $m \to \infty$, the Eqs. (52) and (53) take the values of $C_R = 0$ and $C_T = 1$, however, due to the singularity of the step, this mathematical model not reproduce properly the wave hydrodynamic for different depth ratios in the range $0 < \varepsilon < 1$ for the step case, thus is necessary that for this model the shelf slope have to be finite.

Fig. 2 show the comparison of the present analytical model and the analytical solutions for an infinite step, $m \to \infty$, obtained by Lamb (1932), Eq. (55), Sugimoto et al. (1987), Eq. (56) and the numerical solution given by Pengzhi (2004). For the results shown, in this work we considered a finite shelf slope $m = 15$ which is equivalent to $86.2°$ and not $m \to \infty$ that was because, as was mentioned previously, the step is a highly singular problem, however as can be appreciated the results for the shelf slope proposed have a very good agreement. In the same figure the solution given by Sugimoto et al. (1987) underestimates the reflection coefficient. The present model provides larger values than those obtained Pengzhi (2004); this is because the present model was obtained by potential flow theory and Pengzhi (2004) considered the solution of a turbulent flow model, however for values of ε less than or equal to 0.2, there is a difference of approximately 1.7 %, for values above 0.2 , both solutions are almost identical. The reflection and transmission coefficient have to preserve energy, for instance, if $\varepsilon = 0.5$, the reflection and transmission coefficients are $C_R = 0.17157$ and $C_T = 0.99924$, Fig. 2 and 3, that is $C_R^2 + C_T^2 = 1.027$, which slightly exceed the unit, this may be due to the approximated analytical solution over the shelf, region R1, which is obtained considering only an expansion series to the first order, however the results are in the same magnitude order to those reported by Hsien-Kuo and Jin-Cheng (2007).

$$C_R = \frac{1-\varepsilon^{1/2}}{1+\varepsilon^{1/2}} \tag{55}$$

$$C_R = \frac{1}{4}\left[\sqrt{1+8\left(\frac{1-\varepsilon^{1/2}}{1+\varepsilon^{1/2}}\right)}-1\right]^2 \tag{56}$$

Fig. 3, depicted the reflection and transmission coefficients for different values of the kinematic parameter κ, in this figures the shelf slope change as a function of the parameter ε, the horizontal projection of the shelf L remain constant, the shelf slope $m = (h_2 - h_3)/L$ is expressed as $m = (1-\varepsilon)\gamma$, where $\gamma = h_2 / L$ is constant. In the same figures the coefficients C_R and C_T are evaluated with different values of the parameter $\kappa(= 0.001, 0.01 \text{ and } 0.05)$, the results show that for large values of parameter κ, the reflection coefficient increase and the transmission decrease. In this figures we have include results for the coefficient C_R and C_T for a slope $m = 100$ in order to appreciate the singularity of the infinite step, as can be observed if the slope increase, the solution diverge.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

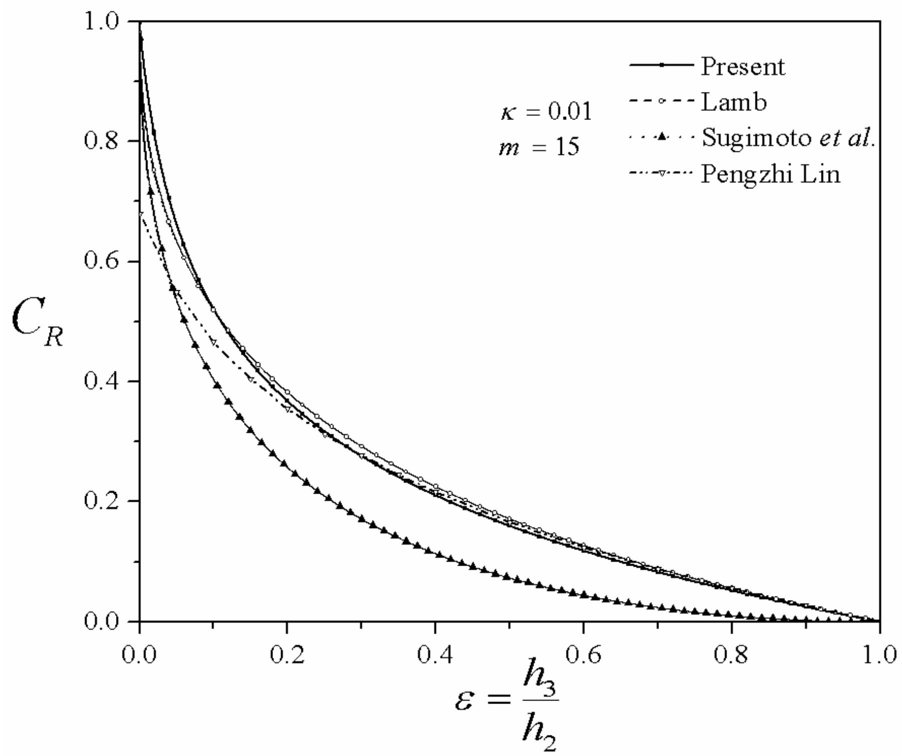

Fig. 2. Wave reflection coefficient for the case of an infinite step using the present model, Lamb, Pengzhi Lin and Sugimoto et al. methods.

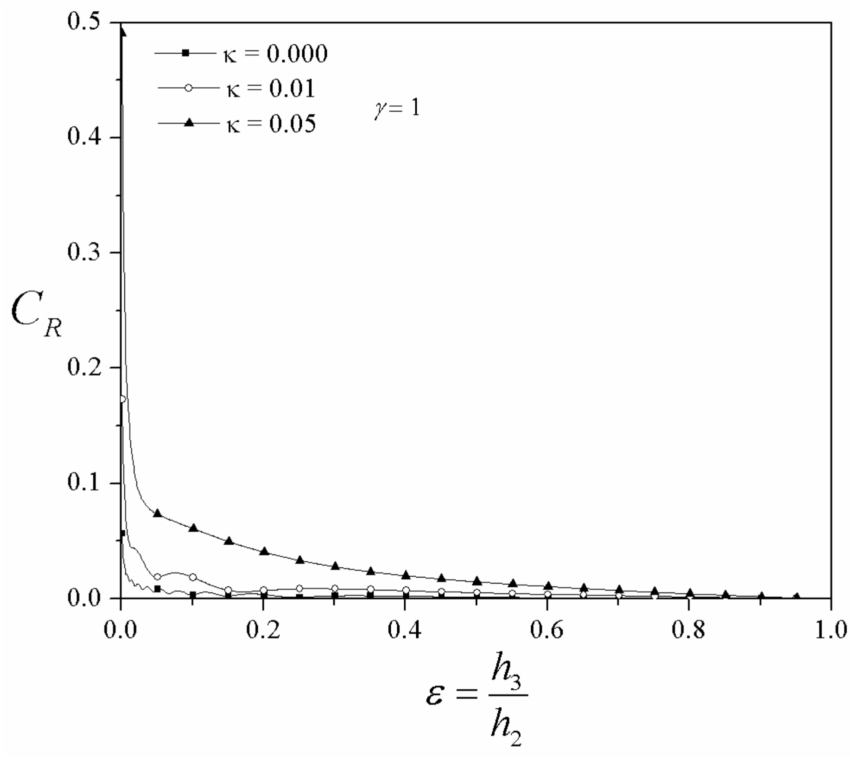

Fig. 3. Wave reflection coefficient for various kinematical parameters $\kappa(= 0.0001, 0.01 \text{ and } 0.05)$ and sloping faces, the horizontal projection of the shelf remain constant.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

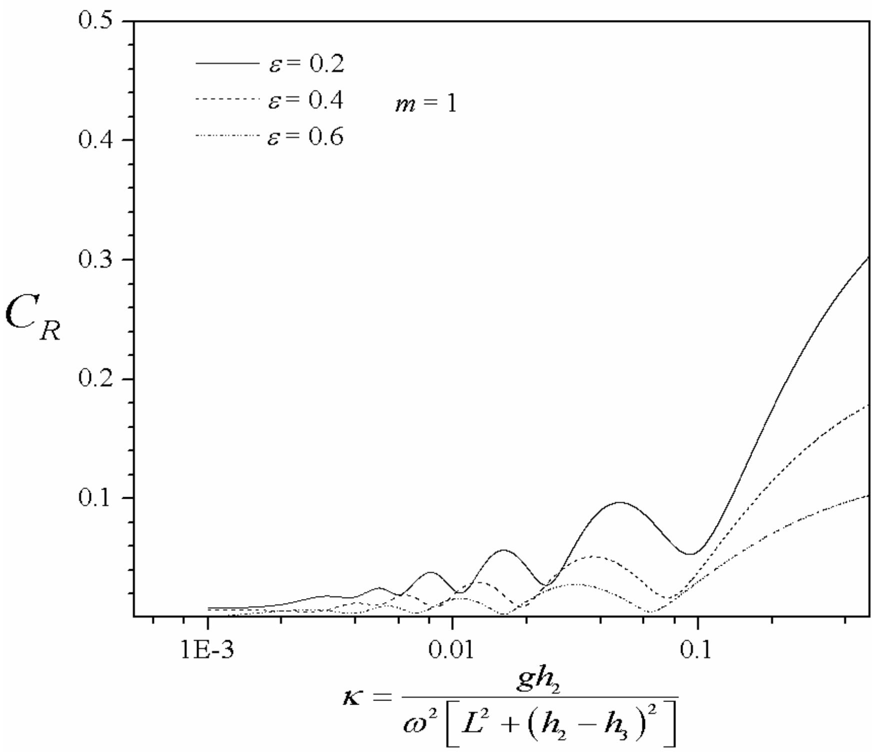

Fig. 4. Wave reflection coefficient as a function of the kinematical parameter κ with values of $\varepsilon(=0.2, 0.4 \text{ and } 0.6)$ and $m = 1$.

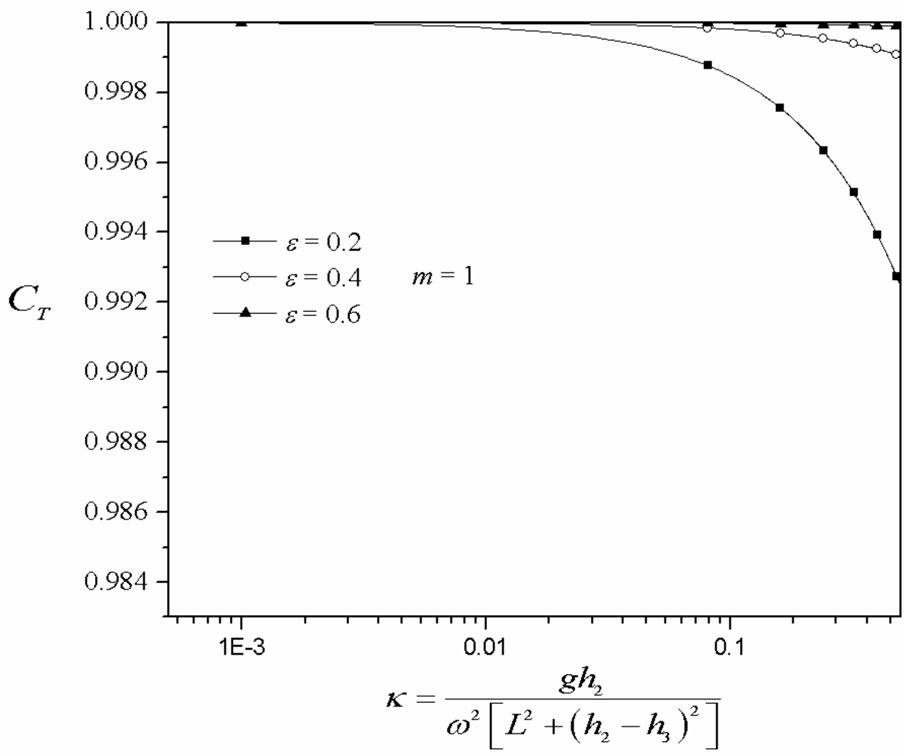

Fig. 5. Wave transmission coefficient as a function of the kinematical parameter κ with values of $\varepsilon(=0.2, 0.4 \text{ and } 0.6)$ and $m = 1$.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

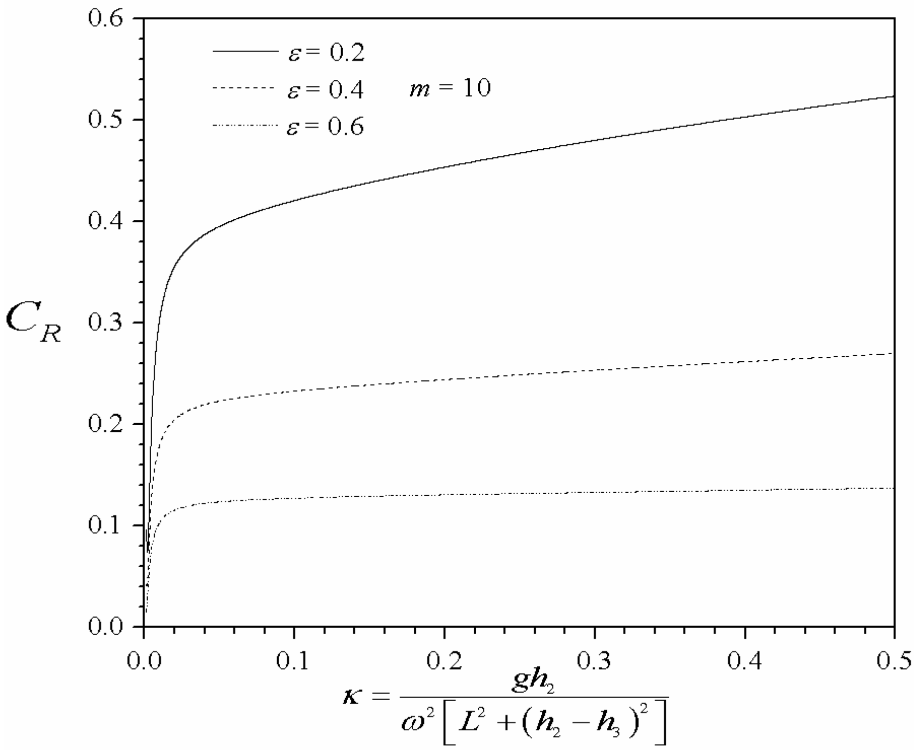

Fig. 6. Wave reflection coefficient as a function of the kinematical parameter κ with values of $\varepsilon(= 0.2, 0.4 \text{ and } 0.6)$ and $m = 10$.

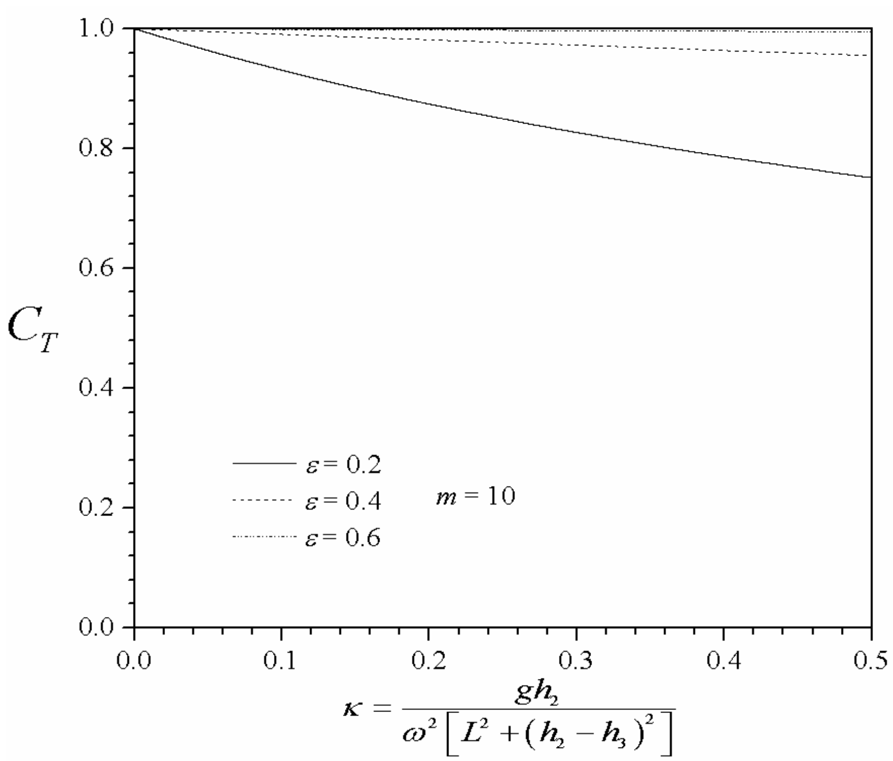

Fig. 7. Wave transmission coefficient as a function of the kinematical parameter κ with values of $\varepsilon(= 0.2, 0.4 \text{ and } 0.6)$ and $m = 10$.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

6 CONCLUSIONS

An analytical model based on the WKB perturbation technique in order to calculate the hydrodynamic response of the reflection coefficient C_R for an ocean wave propagation over a shelf with linear transition has been proposed and solved in a dimensionless form. In the present model the shelf slope can take different values. In particular we have recovered the limits of a flat bottom, $\varepsilon \to 1$, that is $C_R = 0$ and $C_T = 1$.

On the other hand, in order to validate this analytical model, we obtain the reflection and transmission coefficients of waves in the limit of $\varepsilon \ll 1$ and a finite shelf slope $m = 15$, this case tends to the hydrodynamics of an infinite step. Our results were compared with the analytical and numerical solutions for the strict case of a infinite step, reported by Lamb (1932), Pengzhi (2004), Sugimoto et al. (1987). The results were in a good agreement.

The asymptotic analytical model proposed can be used widely in order to analyze in a first approximation, continental shelf effects on the propagation of long waves with large periods and can be extended to analyze different geometries of breakwaters.

ACKNOWLEDGMENTS

This research is financially supported by the research grant number CONACyT 84600 from the Consejo Nacional de Ciencia y Tecnología and project SIP 20113160.

REFERENCE

Aubrey, D. G. and Speer, P. E., A study of non-linear tidal propagation in shallow inlet/estuarine system, Part II: theory. Estuar Coast Shelf Sci, 21, 207-24, 1985.

Baghlani, A., Talebbeydokhti, N. and Abedini, M. J., A shock-capturing model based on flux-vector splitting method in boudary-fitted curvilinear coordinates. *Appl Math Model* 32**,** 249-66, 2008.

Bautista, E. G., Méndez, F., Bautista, O. and Mora, A., Propagation of shallow water waves in an open parabolic channel using the WKB perturbation technique. *Appl Ocean Res*, 33**,** 186-92, 2011.

Bender, C. J. and Dean, R. G., Wave field modification by bathymetric anomalies and resulting shoreline changes: a review with recent results. *Ocean Eng*, 49**,** 125-53, 2003.

Bender, C. M. and Orzag, S. A. (Eds.), *Advanced mathematical methods for scientists and engineers*, New York, McGraw-Hill company, 1978.

Bokhove, O., Flooding and drying in discontinuous Galerkin finite-element discretizations of shallow-water equations. Part 1: one dimension. *J Sci Comput*, 22**,** 47-82, 2005.

Camassa, R., Huang, J. and Lee, L., Integral and integrable algorithms for a nonlinear shallow water wave equation *J Comput phys*, 2006.

Dean, R. G. and Dalrymple, R. A. (Eds.), *Water wave mechanics for engineers and scientists,* London, World Scientific Publishing Co, 2004.

George, D. L., Augmented Riemann solvers for the shallow water equations over variable topography with steady state and inundation. *J Comput phys*, 227**,** 3089-113, 2008.

Goring, D. G. Tsunamis-The propagation of long waves onto a shelf. *California Institute of technology*. Pasadena California, Institute of Technology,1979.

Guinot, V., An aproximation two-dimentional Riemann solver for hyperbolic systems of conservation laws *J Comput phys*, 205**,** 292-314, 2005.

Hsien-kuo, C. and Jin-Cheng, L., Long wave reflection from submerged trapezoidal breakwaters *Ocean Eng*, 34**,** 185-91, 2007.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar

Lamb, H. (Ed.) (1932) *Hydrodynamics* Cambridge (UK), Cambridge University Press.

Läuter, M., Handorf, D. and Dethloff, K, Unsteady analytical solutions of the spherical shallow water equations *J Comput phys,* 210**,** 535-53, 2005.

Li, J. and Jeng, D. S., Note on the propagation of shallow water wave in water of variable depth. *Ocean Eng,* 34**,** 1336-43, 2006.

Losada, M. A., Vidal, C. and Medina, R., Experimental study of the evolution of a solitary wave at an abrupt junction *J Geophys Res,* 94**,** 14557-66, 1989.

Mead, J. L., The shallow water equation in lagrangian coordinates *J Comput phys*, 200**,** 654-69, 2004.

Mei, C. C., Stiassnie, M. and Yue, D. P. (Eds.), *Theory and applications of ocean surface waves. Part 1: Linear aspects. Advance on ocean engineering,* New York (USA), World Scientific Publishing, 2005

Milson, R., Liouville transformation and exactly solvable Schrödinger equations. *Int J Theor Phys* 37**,** 1735-752, 1997.

O'hare, T. J. and Davies, A. G., A new model for surface wave propagation over new model for surface wave propagation over undulating topography *J Coastal Eng,* 18**,** 251-66, 1992.

Ouahsine, A. and Bois, P. A., Mathematical modeling of long ocean waves with a discontinuous density gradient. *J Eng Math* 40**,** 141-58, 2001.

Pengzhi, L., A numerical study of solitary wave interaction with rectangular obstacles *Coastal Eng,* 5**,** 35-51, 2004.

Pengzhi, L. and Liu, H. W., Analytical study of linear long-wave reflection by a two dimensional obstacle of general trapezoidal shape. *J Eng Mech,* 131**,** 822-30, 2005.

Prokof, V. A., State of the art numerical scheme based on the control volume method for modeling turbulent flow sand dam break waves *Power technol Eng,* 36**,** 22-9, 2002.

Rahman, M. (Ed.), *Water waves: relating modern theory to advanced engineering applications.,* Oxford (New York), Clarendon Press, 1995.

Sugimoto, N., Nakajima, N. and Kakutani, T., Edge-layer theory for shallow-water waves over a step-reflection and transmission of a soliton *J.Phys.Soc.Jpn,* 56**,** 1717-30, 1987.

Tae-hwa, J. and Yong-Sik, C., Analytical approach for long wave solution to an arbitrarily varying topography *Coastal Res,* 25**,** 216-23, 2009.

Toro, E. F. (Ed.), *Shock-capturing methods for free-surface shallow flows.,* New York, Wiley, 2001.

Wang, D. J., Katory, M. and Li, Y. S., Analytical and experimental investigation on the hydrodynamic performance of onshore wave-power devices. *Ocean Eng,* 29**,** 871-85, 2002.

Wang, S. K., Hsu, T. W., WENG, W. K. and OU, S. H., A three-point method for estimating wave reflection of obliquely incident waves over a sloping bottom. *Coastal Eng,* 55**,** 125-38, 2008.

Wu, T. Y., Long waves in ocean and coastal waters. *J Eng Mech,* 107**,** 501-22, 1981.

Copyright © 2011 Asociación Argentina de Mecánica Computacional http://www.amcaonline.org.ar