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SOME RESULTS ON THE RANDOM WEAR COEFFICIENT OF ARCHARD MODEL

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Abstract. The most used model for predicting wear is the linear wear law proposed by Archard. A common generalization of Archard's wear law is based on the assumption that the wear rate at any point on the contact surface is proportional to the local contact pressure and the relative sliding velocity. This work focuses on a stochastic modeling of the wear process to take into account the experimental uncertainties in the identification process of the contact-state dependent wear coefficient. The description of the dispersion of the wear coefficient is described by a probability density function, which is performed using the Maximum Entropy Principle using only the information available. Closed-form results for four situations that commonly occur in practice are provided.

1 INTRODUCTION

In mechanical system modeling, uncertainties are present and, to improve the predicability of the models, they should be taken into account. This work discusses uncertainties present in the linear wear law proposed by Archard (Archard, 1953), the most used model for predicting wear. The Archard's wear law is based on the assumption that the wear rate at any point on the contact surface is proportional to the local contact pressure p and the relative sliding velocity υ according to the initial value problem (IVP)

$$\begin{aligned} &\frac{d}{dt}h(t) = p\,\upsilon\,\kappa, \quad t \in (0,T), \\ &h(0) = h_0, \end{aligned} \tag{1}$$

where h_0 is the initial average height, T is the total time, and κ is the contact-state dependent wear coefficient.

In this work we focus on a stochastic modeling of the wear process to take into account the experimental uncertainties in the identification process of the wear coefficient. In view of this, the wear coefficient κ in IVP (1) is treated as a random variable that cannot take negative values. Moreover, the local contact pressure is constant and the relative sliding velocity is considered a time-dependent function along the wear process.

IVP (1) with uncertainty on the wear coefficient has been treated recently in Ávila da Silva Jr and Pintaude (2008). Two cases of uncertainty were studied: random variable or time-dependent stochastic process. In the former, the wear coefficient is modeled as a uniform random variable, while in the latter it is modeled employing a truncated Karhunen-Loève expansion, considering the orthonormal random coefficients as uniform, independently and identically distributed random variables (in our point of view it is difficult to infer what kind of process the wear is in this case). Expectation and covariance functions of the worn height process are obtained.

An approach to deal with IVP (1) is to solve numerically appropriate equations for representative sets of realizations of random variables and to average computed functions. This approach is the so-called Monte Carlo method (see, for example, Fishman, 1996) which has the advantage of applying to a very broad range of both linear and nonlinear problems. The large volume of calculation, the errors in solving the deterministic equations, and the difficulty for generalizing the results may limit the significance of this approach.

The organization of this article is as follows: Section 2 presents the probability density function (pdf) and the joint (two-point) pdf of the wear height stochastic process from the knowledge of the wear coefficient pdf. After, in Section 3, the wear coefficient is considered as uncertain and an approximation of its pdf is deduced from the Maximum Entropy Principle (MEP) using only the a priori information available. Closed-form results for the pdf of $h(t)$, at a fixed t, for four situations that commonly occur in practice are provided. Finally, some conclusions are made in 4.

2 PROBABILITY DENSITY FUNCTION OF THE RANDOM HEIGHT

In this section the pdf and the joint (two-point) pdf of the wear height process are obtained from the knowledge of the wear coefficient pdf, $f_\kappa(q)$. These results will be useful in the next section.

Note that for each realization $\kappa(\omega)$, of the wear coefficient (see, for example, Kloeden and Platen, 1999; Papoulis, 1984, for more details on probability spaces and stochastic processes),

(1) becomes a deterministic IVP whose solution $h(t,\omega)$ can be expressed as

$$h(t,\omega) = h_0 + p\,\kappa(\omega)\int_0^t \upsilon(\tau)\,d\tau, \quad t \in [0,T).$$

Thus, the solution of IVP (1) can be written as

$$h(t) = h_0 + p\,\kappa\,\Upsilon(t), \quad t \in [0,T), \tag{2}$$

where

$$\Upsilon(t) = \int_0^t \upsilon(\tau)\,d\tau.$$

By (2), the cumulative probability function of $h(t)$, $t > 0$, is

$$\begin{aligned} F_h(q;t) &= \mathcal{P}(h(t) \le q) = \mathcal{P}\left(h_0 + p\kappa\Upsilon(t) \le q\right) = \\ &= \mathcal{P}\left(\kappa \le \frac{q-h_0}{p\Upsilon(t)}\right) = F_\kappa\left(\frac{q-h_0}{p\Upsilon(t)}\right), \end{aligned}$$

where $\mathcal{P}$ denotes the probability measure.

Taking the derivative of $F_h(q;t)$ above with respect to q we obtain the following result:

Proposition 2.1. *The density function at a fixed* t, $f_h(q;t)$, *is given by*

$$f_h(q;t) = \frac{1}{p\Upsilon(t)}\,f_\kappa\left(\frac{q-h_0}{p\Upsilon(t)}\right). \tag{3}$$

Corollary 2.1. *The* n*-th moment,* $\mathrm{E}[(h(t))^n]$, $n \in \mathbf{Z}$, $n \ge 1$, *of the solution of (1) is given by*

$$\mathrm{E}[(h(t))^n] = \sum_{j=0}^{n}\binom{n}{j}(p\Upsilon(t))^j\,\mathrm{E}[\kappa^j]\,h_0^{n-j}. \tag{4}$$

In particular, the mean and the variance are given by

$$\mathrm{E}[(h(t))] = p\Upsilon(t)\,\mathrm{E}[\kappa] + h_0 \quad \textit{and} \tag{5}$$

$$\mathrm{Var}[(h(t))] = (p\Upsilon(t))^2\,\mathrm{Var}[\kappa], \tag{6}$$

respectively.

Proof. By (3), it follows that

$$\begin{aligned} \mathrm{E}[(h(t))^n] &= \int_{-\infty}^{+\infty} q^n\,f_h(q;t)\,dq = \frac{1}{p\Upsilon(t)}\int_{-\infty}^{+\infty} q^n\,f_\kappa\left(\frac{q-h_0}{p\Upsilon(t)}\right)\,dq = \\ &= \int_{-\infty}^{+\infty}(p\Upsilon(t)\,\tau + h_0)^n\,f_\kappa(\tau)\;d\tau = \mathrm{E}[(p\Upsilon(t)\,\kappa + h_0)^n] = \\ &= \mathrm{E}\left[\sum_{j=0}^{n}\binom{n}{j}(p\Upsilon(t)\,\kappa)^j\,h_0^{n-j}\right] = \sum_{j=0}^{n}\binom{n}{j}(p\Upsilon(t))^j\,\mathrm{E}[\kappa^j]\,h_0^{n-j}. \end{aligned}$$

□

From (5) and (6), the coefficient of variation of $h(t)$ can be presented as ($h_0 > 0$)

$$\gamma[(h(t))] = \frac{p\Upsilon(t)\,(\mathrm{Var}[\kappa])^{1/2}}{p\Upsilon(t)\,\mathrm{E}[\kappa] + h_0} \leq \frac{(\mathrm{Var}[\kappa])^{1/2}}{\mathrm{E}[\kappa]} = \gamma_\kappa,$$

where γ_κ is the coefficient of variation of κ.

In order to illustrate expression (3) we present next three examples in which κ is a positive real-valued random variable.

Example 2.1. Let κ be a random variable uniformly distributed in $[a, b]$, $\kappa \sim U[a, b]$, $a > 0$. Its density function is

$$f_\kappa(q) = \begin{cases} \dfrac{1}{b-a}, & \text{if } \ a \leq q \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

From (3) it follows that

$$f_h(q;t) = \begin{cases} \dfrac{1}{p\Upsilon(t)(b-a)}, & \text{if } \ ap\Upsilon(t) + h_0 \leq q \leq bp\Upsilon(t) + h_0, \\ 0, & \text{otherwise,} \end{cases}$$

that is, $h(t) \sim U[h_0 + ap\Upsilon(t),\ h_0 + bp\Upsilon(t)]$.

Example 2.2. Let $\kappa \sim G[\alpha, \beta]$ be a gamma random variable with parameters α and β, $\alpha > 0$, $\beta > 0$. Its density function is

$$f_\kappa(q) = \mathbf{1}_{(0,+\infty)}\,\frac{q^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha}\exp\left(-\frac{q}{\beta}\right),$$

where $\mathbf{1}_\mathcal{A}(x) = 1$, if $x \in \mathcal{A}$ and $\mathbf{1}_\mathcal{A}(x) = 0$, if $x \notin \mathcal{A}$; Γ is the Gamma function defined for $\alpha > 0$ as

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}\exp\left(-x\right)dx.$$

From (3) the pdf of $h(t)$ is

$$f_h(q;t) = \mathbf{1}_{(h_0,+\infty)}\,\frac{(q-h_0)^{\alpha-1}}{\Gamma(\alpha)(\beta p\Upsilon(t))^\alpha}\exp\left(-\frac{q-h_0}{\beta p\Upsilon(t)}\right), \tag{7}$$

that is, $h(t) \sim h_0 + G[\alpha,\ \beta p\Upsilon(t)]$. Since $\mathrm{E}[\kappa] = \alpha\beta$ and $\mathrm{Var}[\kappa] = \alpha\beta^2$, it follows directly that $\mathrm{E}[h(t)] = \alpha\beta p\Upsilon(t)$ and $\mathrm{Var}[h(t)] = \alpha(\beta p\Upsilon(t))^2$. The exponential distribution case is obtained by doing $\alpha = 1$ in the above calculations.

Example 2.3. Now let κ be a log-normal random variable, $\kappa = \exp\left(\chi\right)$, $\chi \sim N[\mu, \sigma^2]$. Its density function is given by

$$f_\kappa(q) = \frac{1}{q\sqrt{2\pi}\sigma}\exp\left[-\frac{(\ln\left(q\right) - \mu)^2}{2\sigma^2}\right].$$

Again, from (3) it follows that

$$f_h(q;t) = \frac{1}{q\sqrt{2\pi}p\Upsilon(t)\sigma}\exp\left[-\frac{(\ln\left(q-h_0\right) - \ln\left(p\Upsilon(t)\right) - \mu))^2}{2\sigma^2}\right].$$

2.1 Joint pdf of the random height

Let $h(t_1)$ and $h(t_2)$ be the random solutions of (1), $t_1, t_2 > 0$. As known, second-order properties of a random process can give significant information about the process such as the correlation of $h(t_1)$ and $h(t_2)$, that demands the joint density function, $f_h(q_1, q_2; t_1, t_2)$, of these random variables. The joint cumulative function of $h(t_1)$ and $h(t_2)$ is given by

$$\begin{aligned} F_h(q_1, q_2; t_1, t_2) &= \mathcal{P}(h(t_1) \le q_1,\ h(t_2) \le q_2) = \\ &= \mathcal{P}(h_0 + p\Upsilon(t_1)\,\kappa \le q_1,\ h_0 + p\Upsilon(t_2)\,\kappa \le q_2) = \\ &= \mathcal{P}\left(\kappa \le \frac{q_1 - h_0}{p\Upsilon(t_1)},\ \kappa \le \frac{q_2 - h_0}{p\Upsilon(t_2)}\right) = \\ &= \mathcal{P}\left(\kappa \le \min\left\{\frac{q_1 - h_0}{p\Upsilon(t_1)}, \frac{q_2 - h_0}{p\Upsilon(t_2)}\right\}\right) = F_\kappa\left(\,\varphi(\rho(q_1), \theta(q_2))\,\right), \end{aligned} \tag{8}$$

where $\rho(q_1) = (q_1 - h_0)/p\Upsilon(t_1)$, $\theta(q_2) = (q_2 - h_0)/p\Upsilon(t_2)$, and $\varphi(\rho, \theta) = \min\{\rho, \theta\}$.

We can write the φ - function as

$$\varphi(\rho, \theta) = \min\{\rho, \theta\} = \rho - (\rho - \theta)\, H(\rho - \theta),$$

where H is the Heaviside function (Zauderer, 1983). Also, since $H'(\alpha) = \delta(\alpha)$, the Dirac distribution, the derivatives of φ, in the sense of distributions (Zauderer, 1983), are

$$\frac{\partial \varphi}{\partial \rho} = 1 - H(\rho - \theta) \quad \text{and} \quad \frac{\partial \varphi}{\partial \theta} = H(\rho - \theta).$$

Moreover,

$$\frac{\partial^2 \varphi}{\partial \theta\, \partial \rho} = \delta(\rho - \theta) \quad \text{and} \quad \frac{\partial \varphi}{\partial \rho} . \frac{\partial \varphi}{\partial \theta} = (1 - H(\rho - \theta))\, H(\rho - \theta) = 0.$$

Now we use these expressions to obtain the second-order mixed derivative derivative of (8):

$$\frac{\partial}{\partial q_1} F_h(q_1, q_2; t_1, t_2) = f_\kappa\left(\,\varphi(\rho(q_1), \theta(q_2))\,\right)\, \frac{\partial \varphi}{\partial \rho}(\rho(q_1), \theta(q_2))\, \rho'(q_1),$$

and

$$\begin{aligned} \frac{\partial^2}{\partial q_2\, \partial q_1} F_h(q_1, q_2; t_1, t_2) &= f_\kappa\left(\,\varphi(\rho(q_1), \theta(q_2))\,\right)\, \frac{\partial^2 \varphi}{\partial \theta\, \partial \rho}(\rho(q_1), \theta(q_2))\, \rho'(q_1)\, \theta'(q_2) + \\ &+ f_\kappa'\left(\,\varphi(\rho(q_1), \theta(q_2))\,\right)\, \frac{\partial \varphi}{\partial \rho} . \frac{\partial \varphi}{\partial \rho}(\rho(q_1), \theta(q_2))\, \rho'(q_1)\, \theta'(q_2) = \\ &= f_\kappa\left(\,\varphi(\rho(q_1), \theta(q_2))\,\right)\, \rho'(q_1)\, \theta'(q_2)\, \delta(\rho(q_1) - \theta(q_2)) = \\ &= f_\kappa\left(\rho(q_1)\right)\, \rho'(q_1)\, \theta'(q_2)\, \delta(\rho(q_1) - \theta(q_2)), \end{aligned}$$

since $f_\kappa(\min\{\rho, \theta\})\delta(\rho - \theta) = f_\kappa(\rho)\delta(\rho - \theta) = f_\kappa(\theta)\delta(\rho - \theta)$ (Zauderer, 1983).

With these arguments we have proved the result that follows:

Proposition 2.2. *Let $h(t_1)$ and $h(t_2)$ be the solutions of (1), $t_1, t_2 > 0$. Then, the joint density function of these random variables is given by*

$$f_h(q_1, q_2; t_1, t_2) = \frac{\partial^2}{\partial q_2\, \partial q_1} F_h(q_1, q_2; t_1, t_2) = \frac{1}{p^2 \Upsilon(t_1) \Upsilon(t_2)} f_\kappa\left(\rho(q_1)\right)\, \delta(\rho(q_1) - \theta(q_2)), \tag{9}$$

where $\rho(q_1) = (q_1 - h_0)/p\Upsilon(t_1)$ and $\theta(q_2) = (q_2 - h_0)/p\Upsilon(t_2)$.

Corollary 2.2. *Let $h(t_1)$ and $h(t_2)$ be the solutions of (1), $t_1, t_2 > 0$. Then, the covariance, $\mathrm{Cov}_h(t_1, t_2)$, of these random variables is given by*

$$\mathrm{Cov}_h(t_1, t_2) = p^2 \Upsilon(t_1)\Upsilon(t_2)\,\mathrm{Var}[\kappa]. \tag{10}$$

Proof. Observe that

$$\begin{aligned}
\mathrm{E}[h(t_1)h(t_2)] &= \iint_{\mathbb{R}^2} q_1 q_2\, f_h(q_1, q_2; t_1, t_2)\, dq_1 dq_2 = \\
&= \frac{1}{p^2\Upsilon(t_1)\Upsilon(t_2)} \iint_{\mathbb{R}^2} q_1 q_2\, f_\kappa\left(\rho(q_1)\right)\, \delta(\rho(q_1) - \theta(q_2))\, dq_1 dq_2 = \\
&= \iint_{\mathbb{R}^2} [\rho(q_1)p\Upsilon(t_1) + h_0]\,[\theta(q_2)p\Upsilon(t_2) + h_0]\;\, f_\kappa(\rho(q_1))\, \delta(\rho(q_1) - \theta(q_2))\, d\rho(q_1) d\theta(q_2) = \\
&= \int_{\mathbb{R}} [\theta(q_2)p\Upsilon(t_1) + h_0]\,[\theta(q_2)p\Upsilon(t_2) + h_0]\;\, f_\kappa(\theta(q_2))\, d\theta(q_2) = \\
&= p^2\Upsilon(t_1)\Upsilon(t_2)\,\mathrm{E}[\kappa^2] + p(\Upsilon(t_1) + \Upsilon(t_2))h_0\,\mathrm{E}[\kappa] + h_0^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Cov}_h(t_1, t_2) &= \mathrm{E}[h(t_1)h(t_2)] - \mathrm{E}[h(t_1)]\mathrm{E}[h(t_2)] = \\
&= p^2\Upsilon(t_1)\Upsilon(t_2)\,\mathrm{E}[\kappa^2] + p(\Upsilon(t_1) + \Upsilon(t_2))h_0\,\mathrm{E}[\kappa] + h_0^2 - \\
&- (p\Upsilon(t_1)\mathrm{E}[\kappa] + h_0)(p\Upsilon(t_2)\mathrm{E}[\kappa] + h_0) = \\
&= p^2\Upsilon(t_1)\Upsilon(t_2)\,\left(\mathrm{E}[\kappa^2] - \mathrm{E}[\kappa]^2\right) = p^2\Upsilon(t_1)\Upsilon(t_2)\,\mathrm{Var}[\kappa].
\end{aligned}$$

□

In order to illustrate expression (9) we present next the joint (two-point) pdf of $h(t)$ when κ has a gamma distribution (as in Example 2.2):

$$f_h(q_1, q_2; t_1, t_2) = \mathbf{1}_{(h_0, +\infty)}\, \frac{(q_1 - h_0)^{\alpha - 1}}{\Gamma(\alpha)(\beta p\Upsilon(t_1))^\alpha (p\Upsilon(t_2))}\, \exp\left(-\frac{q_1 - h_0}{\beta p\Upsilon(t_1)}\right)\, \delta(\rho(q_1) - \theta(q_2)),$$

where $\rho(q_1) = (q_1 - h_0)/p\Upsilon(t_1)$ and $\theta(q_2) = (q_2 - h_0)/p\Upsilon(t_2)$.

It is important to observe that expressions (4) and (10) could be calculated directly from (2), but density functions (3) and (9) give a more general statistical understanding of the solution process.

3 PROBABILITY DENSITY OF WEAR GIVEN INFORMATION ABOUT ITS DISTRIBUTION.

In this section, the wear coefficient is considered as uncertain and an approximation of its pdf is deduced from the MEP using only the a priori information available. The construction of the wear pdf is briefly summarized below.

3.1 Maximum Entropy Principle

Considering a real-valued random variable X, associated with the pdf $f_X(x)$, one can define the entropy by

$$S(X) = -\int_{-\infty}^{+\infty} f_X(x)\, \log\left(f_X(x)\right)\, dx.$$

The Maximum Entropy Principle (see Jaynes, 1957a,b; Kapur and Kesavan, 1992; Soize, 2001; Chevalier et al., 2005; Udwadia, 1989, for applications) is a tool that allows the pdf to be constructed by searching the maximum of $S(X)$ under the constraint of the available information. It gives a pdf that maximizes the uncertainty and is compatible with the known information, that is it does not violate physical principles. Example of informations are, the support of the pdf, the mean value, the standard deviation or higher moments. A Lagrange multiplier λ_i will be associated to each constraint defined by the available information. These constraints are written in the form:

$$\mathrm{E}[g_i(X)] = \int_{-\infty}^{+\infty} g_i(x) f_X(x) dx = f_i, \quad i = 1, 2, ..., m, \tag{11}$$

where $g_i(X)$ are given functions. For instance, if $g_i(x) = x$, f_i is the mean value of X. It can be shown that the multipliers λ_i are obtained by minimizing the strictly convex function H defined by

$$H(\lambda_0, \lambda_1, ..., \lambda_m) = \lambda_0 + \sum_{i=1}^{m} f_i \lambda_i + \int_{-\infty}^{+\infty} \mathbf{1}_{[a,b]}(x)\ \exp\left(-\lambda_0 - \sum_{i=1}^{m} \lambda_i g_i(x)\right) dx,$$

$[a, b]$ denoting the support of the pdf of random variable X. The pdf expression is then given by

$$f_X(x) = \mathbf{1}_{[a,b]}(x)\ \exp\left(-\lambda_0 - \sum_{i=1}^{m} \lambda_i g_i(x)\right). \tag{12}$$

In the next paragraph, we will construct pdf for different sets of available information (Udwadia, 1989; Wragg and Dowson, 1970; Dowson and Wragg, 1973). Thus, closed-form results for the pdf of $h(t)$, at a fixed t, for these situations will be provided.

3.2 Probability density function for the random wear coefficient

We now consider four situations for the fluctuation of wear coefficient that commonly occur in practice: (i) κ is known to lie between 0 and b, where we assume that $0 < b < +\infty$; (ii) κ is known to lie between 0 and b, $0 < b < +\infty$, and its mean is know to be m_1; (iii) κ is known to be positive and its mean is know to be m_1; (iv) κ is known to be positive, its mean is m_1, and its variance is known to be σ^2, that is, its finite second moment m_2 is known.

Case (i). The wear coefficient κ is known to lie in the finite range 0 to b. Using (12) we find that the maximally unpresumptive density of κ is simply a constant. Noting that the area under de density curve is unity we have

$$f_\kappa(x) = \mathbf{1}_{(0,b)}(x)\ \frac{1}{b}, \tag{13}$$

that is, κ is uniform between 0 and b. From Example 2.1 the pdf of $h(t)$ is

$$f_h(q;t) = \begin{cases} \dfrac{1}{bp\Upsilon(t)}, & \text{if } h_0 < q < h_0 + bp\Upsilon(t), \\ 0, & \text{otherwise,} \end{cases}$$

that is, $h(t) \sim U[h_0,\ h_0 + bp\Upsilon(t)]$.

Case (ii). The wear coefficient κ is known to lie in the finite range 0 to b and its mean is m_1. That is, the available information is:

$$\mathrm{supp}(f_\kappa) = (0, b) \quad \text{and} \quad m_1 = \mathrm{E}[\kappa].$$

If we use (12) with $m = 1$, the density function of κ becomes

$$f_\kappa(x) = \begin{cases} \lambda_0 \exp(\lambda_1 x), & \text{if } 0 < x < b, \\ 0, & \text{otherwise}, \end{cases}$$

where λ_0 is a positive constant. From (11), using the relations

$$\int_0^b f_\kappa(x)dx = 1 \qquad \text{and} \qquad \int_0^b x f_\kappa(x)dx = m_1, \tag{14}$$

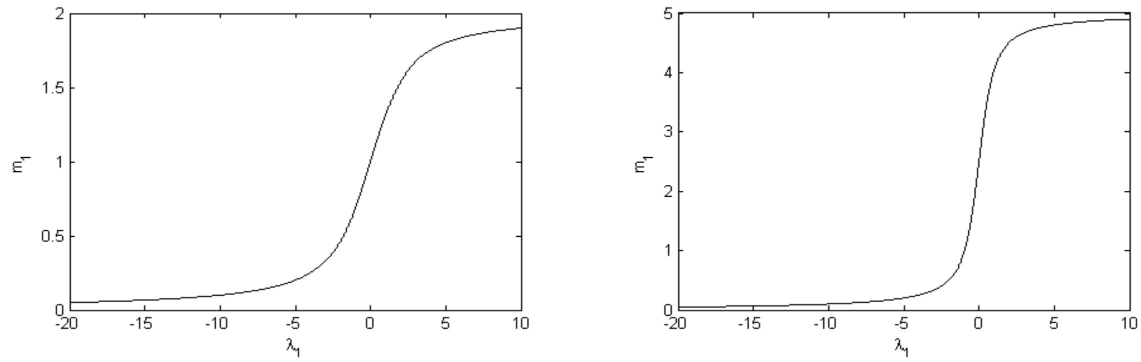
we obtain

$$\lambda_0 = \frac{\lambda_1}{\exp(\lambda_1 b) - 1} \qquad \text{and} \qquad m_1 = \frac{b}{1 - \exp(-\lambda_1 b)} - \frac{1}{\lambda_1}. \tag{15}$$

From (15) it is possible to show that

$$\lim_{\lambda_1 \to -\infty} m_1(\lambda_1) = 0, \quad \lim_{\lambda_1 \to +\infty} m_1(\lambda_1) = b, \quad \lim_{\lambda_1 \to 0} m_1(\lambda_1) = \frac{b}{2}, \quad \text{and}$$

$m_1'(\lambda_1) > 0$ for all λ_1, that is, m_1 seen as a function of λ_1 is a monotonically increasing function. Figure 1 illustrates m_1 as a function of λ_1 for $b = 2$ and $b = 5$.



(a) (b)

Figure 1: Illustration of function $m_1(\lambda_1)$, $b = 2$ (a) and $b = 5$ (b).

From relation (15) and the above limits we see that when $m_1 \to b/2$, $\lambda_1 \to 0$ and $\lambda_0 \to 1/b$. Thus we obtain a uniform distribution identical to that given by (13). Also, for $m_1 \to 0^+$ we have $\lambda_1 \to -\infty$, and for $m_1 \to b^-$ we have $\lambda_1 \to +\infty$ (in both cases the density tends toward delta distributions). For other values of m_1, the corresponding values of λ_1 can be found by inverting (numerically) m_1 in (15). In this case, we arrives at the following density function for κ:

$$f_\kappa(x) = \begin{cases} \dfrac{\lambda_1 \exp(\lambda_1 x)}{\exp(\lambda_1 x) - 1}, & \text{if } 0 < x < b, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

We observe that for $m_1 < b/2$ the value of λ_1 is always negative and the resulting probability density above is a truncated exponential distribution. From (3) and (16) the pdf of $h(t)$ is given by

$$f_h(q;t) = \begin{cases} \dfrac{\lambda_1}{p\Upsilon(t)\left[1 - \exp\left(-\lambda_1 \dfrac{q - h_0}{p\Upsilon(t)}\right)\right]}, & \text{if } h_0 < q < h_0 + bp\Upsilon(t), \\ 0, & \text{otherwise.} \end{cases}$$

Case (iii). The wear coefficient κ is known to be positive and its mean is know to be m_1. That is, the available information is:

$$\text{supp}(f_\kappa) = (0, +\infty) \quad \text{and} \quad m_1 = \text{E}[\kappa].$$

Using (12) with $m = 1$, the density of κ becomes

$$f_\kappa(x) = \begin{cases} \lambda_0 \exp(\lambda_1 x), & \text{if } 0 < x < +\infty, \\ 0, & \text{otherwise,} \end{cases}$$

where λ_0 is a positive constant. As in (14) with $b = +\infty$ we obtain $\lambda_0 = -\lambda_1$ and $m_1 = \lambda_0/\lambda_1^2$, that is, $\lambda_0 = 1/m_1$ and $\lambda_1 = -1/m_1$. Thus, κ is exponentially distributed with density function given by

$$f_\kappa(x) = \mathbf{1}_{(0,+\infty)}(x)\, \frac{1}{m_1} \exp\left(-\frac{x}{m_1}\right).$$

Since, in this case, $\kappa \sim G[1, m_1]$ (see Example 2.2) it follows from (7) that the density of $h(t)$ is given by

$$f_h(q;t) = \mathbf{1}_{(h_0,+\infty)}\, \frac{1}{m_1 p\Upsilon(t)} \exp\left(-\frac{q - h_0}{m_1 p\Upsilon(t)}\right),$$

that is, $h(t) \sim h_0 + G[1, m_1 p\Upsilon(t)]$. Moreover, it follows directly that $\text{E}[h(t)] = m_1 p\Upsilon(t)$ and $\text{Var}[h(t)] = (m_1 p\Upsilon(t))^2$.

Case (iv). The wear coefficient κ is known to be positive, its mean is m_1, and its finite variance is known to be σ^2. That is, the available information is:

$$\text{supp}(f_\kappa) = (0, +\infty), \quad m_1 = \text{E}[\kappa], \quad \text{and} \quad m_2 = \sigma^2 + m_1^2 < +\infty.$$

According to (12), with $m = 2$, the pdf of κ can be presented as

$$f_\kappa(x) = \mathbf{1}_{(0,+\infty)}(x)\, \lambda_0 \, \exp\left(-(\lambda_1 x + \lambda_2)^2\right).$$

The multipliers λ_0, λ_1 and λ_2 must satisfy the following conditions ($\lambda_1 > 0$):

$$1 = \lambda_0 \int_0^{+\infty} \exp\left(-(\lambda_1 x + \lambda_2)^2\right) dx,$$
$$m_1 = \lambda_0 \int_0^{+\infty} x \, \exp\left(-(\lambda_1 x + \lambda_2)^2\right) dx, \quad \text{and}$$
$$\sigma^2 + m_1^2 = \lambda_0 \int_0^{+\infty} x^2 \, \exp\left(-(\lambda_1 x + \lambda_2)^2\right) dx.$$

Or, after algebraic manipulations,

$$\lambda_0 = \frac{2\lambda_1}{\sqrt{\pi}\,\mathrm{erfc}\,(\lambda_2)},$$
$$m_1 = \frac{\exp\left(-\lambda_2^2\right)}{\lambda_1\sqrt{\pi}\,\mathrm{erfc}\,(\lambda_2)} - \frac{\lambda_2}{\lambda_1}, \quad \text{and} \tag{17}$$
$$\sigma^2 + m_1^2 = \frac{1}{2\lambda_1^2} + \frac{\lambda_2^2}{\lambda_1^2} - \frac{\lambda_2}{\lambda_1^2}\frac{\exp\left(-\lambda_2^2\right)}{\lambda_1\sqrt{\pi}\,\mathrm{erfc}\,(\lambda_2)} = \frac{1}{2\lambda_1^2} - \frac{\lambda_2}{\lambda_1} m_1. \tag{18}$$

From (18) it follows

$$\lambda_1 m_1 = \frac{-\lambda_2 \pm \sqrt{\lambda_2^2 + 2(\gamma^2+1)}}{2(\gamma^2+1)}.$$

where $\gamma = \sigma/m_1$ is the coefficient of variation.

Since $\lambda_1 > 0$, and using (17), we can write

$$\frac{\exp\left(-\lambda_2^2\right)}{\sqrt{\pi}\,\mathrm{erfc}\,(\lambda_2)} - \lambda_2 = \lambda_1 m_1 = \frac{-\lambda_2 + \sqrt{\lambda_2^2 + 2(\gamma^2+1)}}{2(\gamma^2+1)}.$$

Denoting $\Delta(\lambda_2) = \exp\left(-\lambda_2^2\right)/(\sqrt{\pi}\,\mathrm{erfc}\,(\lambda_2))$ we can express γ as

$$\gamma(\lambda_2) = \frac{1}{\sqrt{2}\,[\Delta(\lambda_2) - \lambda_2]}\left[1 - 2\left(\Delta(\lambda_2) - \lambda_2\right)\lambda_2 - 2\left(\Delta(\lambda_2) - \lambda_2\right)^2\right]^{1/2}.$$

Figure 2 illustrates γ as a function of λ_2. Also, it is important to note that

$$\lim_{\lambda_2 \to +\infty} \frac{\Delta(\lambda_2)}{\lambda_2} = 1, \quad \lim_{\lambda_2 \to -\infty} \Delta(\lambda_2) = 0, \quad \Delta(\lambda_2) \text{ is monotonically increasing},$$
$$\lim_{\lambda_2 \to -\infty} \gamma(\lambda_2) = 0, \quad \text{and} \quad \lim_{\lambda_2 \to +\infty} \gamma(\lambda_2) = 1.$$

Thus, since $\gamma(-7.07) \simeq 0.1$, $\gamma(-3.54) \simeq 0.2$, $\Delta(-7.07) \simeq 5.52\times 10^{-23}$, and $\Delta(-3.54) \simeq 1.01 \times 10^{-6}$, we can rewrite (17) and (18), for $\gamma < 0.2$, as

$$m_1 \simeq -\frac{\lambda_2}{\lambda_1} \quad \text{and} \quad \sigma^2 + m_1^2 = \frac{1}{2\lambda_1^2} - \frac{\lambda_2}{\lambda_1} m_1,$$

that is,

$$\lambda_1 \simeq \frac{1}{\sqrt{2}\sigma} \quad \text{and} \quad \lambda_2 \simeq -\frac{m_1}{\sqrt{2}\sigma}.$$

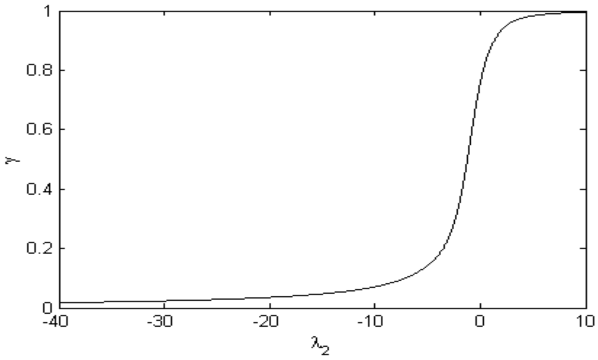


Figure 2: Illustration of γ as a function of λ_2.

Thus, we find that for $\gamma < 0.2$ the resulting pdf is given (well-approximated) by the truncated Gaussian distribution

$$f_\kappa(x) = \mathbf{1}_{(0,+\infty)}(x)\,\frac{2}{\sqrt{2\pi}\sigma\,\mathrm{erfc}\left(-\frac{1}{\sqrt{2}\,\gamma}\right)}\,\exp\left(-\frac{(x-m_1)^2}{2\sigma^2}\right). \tag{19}$$

From (3) and (19) the pdf of $h(t)$, in this case, is

$$f_h(q;t) = \mathbf{1}_{(h_0,+\infty)}(x)\,\frac{2}{\sqrt{2\pi}\sigma p \Upsilon(t)\,\mathrm{erfc}\left(-\frac{1}{\sqrt{2}\,\gamma}\right)}\,\exp\left(-\frac{[q-h_0-m_1 p\Upsilon(t)]^2}{2(\sigma p\Upsilon(t))^2}\right).$$

4 CONCLUSIONS

This paper constructed, using the MEP, several pdf of the wear coefficient κ for different sets of information. Then the pdf of the wear at a time t, $h(t)$, is explicitly derived. Considering the wear a second-order process, the probabilistic characterization of it is also explicitly constructed.

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