Asociación Argentina de Mecánica Computacional

Mecánica Computacional Vol XXXII, págs. 1689-1700 (artículo completo)
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Mendoza, Argentina, 19-22 Noviembre 2013

RESULTS ON THE SIMULTANEOUS USE OF CLASSICAL TIKHONOV-PHILLIPS AND BOUNDED-VARIATION REGULARIZATION METHODS FOR INVERSE ILL-POSED PROBLEMS

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Keywords: Inverse problem, ill-posedness, regularization, Tikhonov-Phillips, bounded variation.

Abstract. Several generalizations of the traditional Tikhonov-Phillips regularization method for inverse ill-posed problems have been proposed during the last two decades. Many of these generalizations are based upon inducing stability throughout the use of different penalizers which allow the capturing of diverse properties of the exact solution (e.g. edges, discontinuities, borders, etc.). However, in some problems in which it is known that the regularity of the exact solution is heterogeneous and/or anisotropic, it is reasonable to think that a much better option could be the simultaneous use of two or more penalizers of different nature. Such is the case, for instance, in some image restoration problems in which preservation of edges, borders or discontinuities is an important matter. In this work we present some new results on the simultaneous use of penalizers of L^2 and of bounded-variation (BV) type. For particular cases, existence and uniqueness results are shown. Open problems are discussed and some results in applications to signal and image restoration problems are presented.

1 INTRODUCTION

We consider the general problem of finding u in an equation of the form

$$Tu = v, \tag{1}$$

where $T : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator between two infinite dimensional Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, the range of T is non-closed and v is the data, which is supposed to be known, perhaps with a certain degree of error. It is well known that under these hypotheses problem (1) is ill-posed and it must be regularized before any attempt is made to approximate its solutions (Engl et al. (1996)). The most usual way of regularizing a problem is by means of the use of the *Tikhonov-Phillips regularization method* whose general formulation can be given within the context of an unconstrained optimization problem. In fact, given an appropriate penalizer $W(u)$ with domain $\mathcal{D} \subset \mathcal{X}$, the regularized solution obtained by the Tikhonov-Phillips method and such a penalizer, is the minimizer u_α (provided it exits), over $\mathcal{D}$, of the functional

$$J_{\alpha,W}(u) = \|Tu - v\|^2 + \alpha W(u), \tag{2}$$

where α is a positive constant called regularization parameter. For general penalizers W, sufficient conditions guaranteeing existence, uniqueness and weak and strong stability of the minimizers under different types of perturbations, where found in Mazzieri et al. (2012).

Each choice of an admissible penalizer W originates a different regularization method producing a particular regularized solution possessing particular properties. Thus, for instance, the choice of $W(u) = \|u\|^2$ gives raise to the classical Tikhonov-Phillips method of order zero producing always smooth regularized approximations which approximate, as $\alpha \to 0^+$, the best approximate solution (i.e. the least squares solution of minimum norm) of problem (1) (see Engl et al. (1996)). Similarly, the choice of $W(u) = \|u\|_{\mathrm{BV}}$ (where $\|\cdot\|_{\mathrm{BV}}$ denotes the total variation norm) results in the so called "bounded variation regularization method" (Acar and Vogel (1994), Rudin et al. (1992)). The use of this penalizer is very appropriate when preserving discontinuities, borders or edges is an important matter. The method, however, has as a drawback that it tends to produce piecewise constant approximations and therefore, it will likely be highly inappropriate near regions where the exact solution is smooth (Chambolle and Lions (1997)) producing the so called "staircase effect".

In certain types of problems, particularly in those in which it is known that the regularity of the exact solution is heterogeneous or anisotropic, it is reasonable to think that using and spatially adapting two or more penalizers of different nature could be more convenient. During the last 15 years several regularization methods have been developed in light of this simple reasoning. Thus, for instance, in 1997 Blomgren *et al.* (Blomgren et al. (1997)) proposed the use of the following penalizer, by using the variable L^p spaces:

$$W(u) = \int_\Omega |\nabla u|^{p(|\nabla u|)} dx, \tag{3}$$

where $\lim_{r\to 0} p(r) = 2$, $\lim_{r\to\infty} p(r) = 1$ and p is a decreasing function. Thus, in regions where the modulus of the gradient of u is small the penalizer is approximately equal to $\||\nabla u|\|^2_{L^2(\Omega)}$ corresponding to a zero-order Tikhonov-Phillips method (appropriate for restoration in smooth regions). On the other hand, when the modulus of the gradient of u is large, the penalizer resembles the bounded variation seminorm $\||\nabla u|\|_{L^1(\Omega)}$, whose use, as mentioned earlier, is highly appropriate for border detection purposes. Although this model for W is quite reasonable, proving basic properties of the corresponding generalized Tikhonov-Phillips functional turns out to be quite difficult. A different way of combining these two methods was proposed by Chambolle and Lions (Chambolle and Lions (1997)). They suggested the use of:

$$W_\beta(u) = \int_{|\nabla u|\leq\beta} |\nabla u|^2\, dx + \int_{|\nabla u|>\beta} |\nabla u|\, dx,$$

where $\beta > 0$ is a given threshold. Thus, in regions where borders are more likely to be present ($|\nabla u| > \beta$), penalization is made with the bounded variation seminorm while a standard order-one Tikhonov-Phillips method is used otherwise. This model was shown to be successful in restoring images possessing regions with homogeneous intensity separated by borders. However, in the case of images with non-uniform or highly degraded intensities, the model is extremely sensitive to the choice of the threshold β. More recently penalizers of the form

$$W(u) = \int_\Omega |\nabla u|^{p(x)} dx, \tag{4}$$

for certain functions p with range in $[1, 2]$, were studied in Chen et al. (2006) and Li et al. (2010). It is timely to point out here that all previously mentioned results work only for the case of denoising, i.e. for the case $T = id$.

In this work we propose the use of a model for general restoration problems, which combines, in an appropriate way, the penalizers corresponding to zero-order Tikhonov-Phillips method and the bounded variation seminorm. Although several mathematical issues for this model still remain open, as we shall see in Section 5, its use in some signal and image restoration problems has proved to be very promising. The purpose of this article is to introduce the model, prove some theorems regarding the existence of the corresponding regularized solutions, and present a few results on their application to some signal and image restoration problems.

2 PRELIMINARIES

From now on Ω will denote a convex region in $\mathbb{R}^n$, $n = 1, 2, 3$, whose boundary $\delta\Omega$ is Lipschitz continuous. The following Theorem, whose proof can be found in Acar and Vogel (1994) (Theorem 3.1), guarantees the well-posedness of the unconstrained minimization problem

$$u^* \;=\; \min_{u\in L^p(\Omega)} J(u). \tag{5}$$

Theorem 2.1 *Let J a functional BV-coercive defined on $L^p(\Omega)$. If $1 \leq p < \frac{n}{n-1}$ and J is lower semicontinuous, then the problem (5) has a solution. If $p = \frac{n}{n-1}$, $n \geq 2$ and in addition J is weakly lower semicontinuous, then a solutions also exists. In either case, the solution is unique if J is strictly convex.*

The following theorem, whose proof can also be found in Acar and Vogel (1994) (Theorem 4.1), focuses on the existence and uniqueness of minimizers of functionals of the form

$$J(u) \;=\; \|Tu - v\|^2 + \alpha J_0(u), \tag{6}$$

where $\alpha > 0$ and $J_0(u)$ denotes the bounded variation seminorm.

Theorem 2.2 *Suppose that p satisfies the restrictions of Theorem 2.1 and $T\chi_\Omega \neq 0$. Then the functional (6) has a minimizer.*

Note here that (6) is a particular case of (2) with $W(u) = J_0(u)$. The following theorem, whose proof can be found in Mazzieri et al. (2012), gives conditions guaranteeing existence and uniqueness of minimizers of (2) for general penalizers $W(u)$. This Theorem will be very important for our main results in the next section.

Theorem 2.3 *(Existence and uniqueness) Let $\mathcal{X}$, $\mathcal{Y}$ be normed vector spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $y \in \mathcal{Y}$, $\mathcal{D} \subset \mathcal{X}$ a convex set and $W : \mathcal{D} \longrightarrow \mathbb{R}$ a functional bounded from below, W-subsequentially weakly lower semicontinuous, and such that W-bounded sets are relatively weakly compact in $\mathcal{X}$. More precisely, suppose that W satisfies the following hypotheses:*

- *(H1): $\exists\, \gamma \geq 0$ such that $W(x) \geq -\gamma \quad \forall\, x \in \mathcal{D}$.*
- *(H2): for every W-bounded sequence $\{x_n\} \subset \mathcal{D}$ such that $x_n \overset{w}{\to} x \in \mathcal{D}$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $W(x) \leq \liminf_{j\to\infty} W(x_{n_j})$.*
- *(H3): for every W-bounded sequence $\{x_n\} \subset \mathcal{D}$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and $x \in \mathcal{D}$ such that $x_{n_j} \overset{w}{\to} x$.*

Then the functional $J_{W,\alpha}(x) \doteq \|Tx - y\|^2 + \alpha W(x)$ has a global minimizer. If moreover W is convex and T is injective or if W is strictly convex, then such a minimizer is unique.

3 MAIN RESULTS

In this section we will state our main results concerning existence and uniqueness of minimizers of particular generalized Tikhonov-Phillips functionals with combines L^2-BV penalizers. Due to brevity and since complete proof of these results will appear in a forthcoming paper, we will not include all proofs here, limiting our discussion only to those considred more relevant. In what follows, Ω shall denote a bounded open convex subset of $\mathbb{R}^n$ with Lipschitz boundary and $\theta : \Omega \to [0, 1]$ a measurable function.

Definition 3.1 *We define the functional $W_{0,\theta}(u)$ by*

$$W_{0,\theta}(u) \doteq \sup_{\vec{\nu}\in\mathcal{V}_\theta} \int_\Omega -u\, \text{div}(\theta\vec{\nu})\, dx, \qquad u \text{ measurable,} \tag{7}$$

where $\mathcal{V}_\theta \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^n$ such that $\theta\vec{\nu} \in C_0^1(\Omega)$ and $|\vec{\nu}(x)| \leq 1\, \forall\, x \in \Omega\}$.

It is not difficult to prove the following two lemmas.

Lemma 3.2 *If u and θ belong to $C^1(\Omega)$ then $W_{0,\theta}(u) = \|\theta\, |\nabla u|\, \|_{L^1(\Omega)}$.*

Observation: From de density of $C^1(\Omega)$ in $W^{1,1}(\Omega)$ it follows that Lemma 3.2 holds for u, $\theta \in W^{1,1}(\Omega)$.

Lemma 3.3 *The functional $W_{0,\theta}$ defined by (7) is weakly lower semicontinuos with respect to the L^p topology, $\forall\, p \in [1, \infty)$.*

Theorem 3.4 *Let $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a normed vector space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $v \in \mathcal{Y}$, α_1, α_2 positive constants and J_θ the functional defined by*

$$J_\theta(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2\, W_{0,\theta}(u), \qquad u \in L^2(\Omega). \tag{8}$$

If there exists $\varepsilon_2 \in \mathbb{R}$, such that $0 \leq \theta(x) \leq \varepsilon_2 < 1$ a.e. $x \in \Omega$, then the functional (8) has a unique global minimizer $u^* \in L^2(\Omega)$. If moreover $\theta \in C^1(\Omega)$ and there exists $\varepsilon_1 \in \mathbb{R}$ such that $0 < \varepsilon_1 \leq \theta(x)$ a.e. $x \in \Omega$, then $u^* \in BV(\Omega)$.

Proof: Due to Theorem 2.3 it is sufficient to show that the functional

$$W(u) \doteq \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2\, W_{0,\theta}(u),\; u \in L^2(\Omega)$$

satisfies hypothesis *(H1)*, *(H2)* and *(H3)*. Clearly *(H1)* holds with $\gamma = 0$.

To prove (*H2*) let $\{u_n\} \subset L^2(\Omega)$ such that $u_n \rightharpoonup u \in L^2(\Omega)$ and $W(u_n) \leq c < \infty$. We want to show that $W(u) \leq \liminf_{n\to\infty} W(u_n)$. Since $\sqrt{1-\theta} \in L^\infty(\Omega)$ one has $\sqrt{1-\theta}\, u_n \rightharpoonup \sqrt{1-\theta}\, u$.

The condition $\theta(x) \leq \varepsilon_2 < 1$ a.e. $x \in \Omega$ clearly implies that $\|\sqrt{1-\theta}\ u\|_{L^2(\Omega)}$ is a norm. Then, from the weak lower semicontinuity of $\|\sqrt{1-\theta}\ u\|^2_{L^2(\Omega)}$ it follows that

$$\|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\sqrt{1-\theta}\, u_n\|^2_{L^2(\Omega)}. \tag{9}$$

On the other hand, from the weak lower semicontinuity of $W_{0,\theta}$ in $L^2(\Omega)$ (Lemma 3.3) it follows that

$$W_{0,\theta}(u) \leq \liminf_{n\to\infty} W_{0,\theta}(u_n). \tag{10}$$

From (9) and (10) we conclude that

$$\begin{aligned} W(u) &= \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2\, W_{0,\theta}(u) \\ &\leq \alpha_1 \liminf_{n\to\infty} \|\sqrt{1-\theta}\, u_n\|^2_{L^2(\Omega)} + \alpha_2 \liminf_{n\to\infty} W_{0,\theta}(u_n) \\ &\leq \liminf_{n\to\infty} \left(\alpha_1 \|\sqrt{1-\theta}\, u_n\|^2_{L^2(\Omega)} + \alpha_2 W_{0,\theta}(u_n)\right) \\ &= \liminf_{n\to\infty} W(u_n), \end{aligned}$$

what proves (*H2*). To prove (*H3*) let $\{u_n\} \subset L^2(\Omega)$ be such that $W(u_n) \leq c < \infty,\ \forall n$. We want to show that there exist $\{u_{n_j}\} \subset \{u_n\}$ and $u \in L^2(\Omega)$ such that $u_{n_j} \rightharpoonup u$. For this note that

$$(1-\varepsilon_2)\|u_n\|^2_{L^2(\Omega)} \leq \|\sqrt{1-\theta}\, u_n\|^2_{L^2(\Omega)} \leq W(u_n) \leq c. \tag{11}$$

Thus the existence of a weakly convergent subsequence follows from the boundedness of $\{u_n\}$ in $L^2(\Omega)$. Hence, by Theorem 2.3, $J_\theta(u)$ given by (8) has a global minimizer $u^* \in L^2(\Omega)$. The condition $\theta(x) \leq \varepsilon_2 < 1$ a.e. $x \in \Omega$ clearly implies the strict convexity of J_θ and therefore the uniqueness of the global minimizer.

For the second part, assume further that $\theta \in C^1(\Omega)$ and there exists $\varepsilon_1 > 0$ such that $\theta(x) \geq \varepsilon_1$ a.e. $x \in \Omega$. Following the proof of Theorem 5.1 in Mazzieri et al. (2012), it suffices to show that under this additional hypothesis the weak limit u in *(H3)* above belongs to $BV(\Omega)$. For this note that, since $W(u_n)$ is uniformly bounded, from (11) it follows that there exist $K < \infty$ such that

$$\|u_n\|_{L^1(\Omega)} \leq K \quad \forall\, n. \tag{12}$$

Also, by Lemma 3.2, if $u \in C^1(\Omega)$ then

$$W_{0,\theta}(u) = \sup_{\vec{\nu}\in\mathcal{V}_\theta} \int_\Omega -u\, \mathrm{div}(\theta\vec{\nu})\, dx \;=\; \|\theta|\nabla u|\|_{L^1(\Omega)} \geq \varepsilon_1 \||\nabla u|\|_{L^1(\Omega)} = \varepsilon_1 J_0(u). \tag{13}$$

Using the density of $C^1(\Omega)$ in $L^2(\Omega)$ it follows that $W_{0,\theta}(u) \geq \varepsilon_1 J_0(u)\ \forall\, u \in L^2(\Omega)$. Thus from (12) and (13) it follows that

$$\|u_n\|_{BV(\Omega)} = \|u_n\|_{L^1(\Omega)} + J_0(u_n) \leq c < \infty\ \forall n.$$

Hence the fact that the weak limit in (*H3*) is in BV(Ω) follows from the compact embedding of BV(Ω) in to $L^2(\Omega)$. This result is an extension of the Rellich-Kondrachov Theorem and can be found, for example, in (Adams (1975), Attouch et al. (2006)). ■

Remark 3.5 *Note that if $\theta(x) = 0 \ \ \forall x \in \Omega$, then J_θ as defined in (8) is the classical Tikhonov-Phillips functional of order zero. On the other hand, if $\theta(x) = 1 \ \ \forall x \in \Omega$ then J_θ has a global minimizer provided that $T\chi_\Omega \neq 0$. If moreover T is injective then such a global minimizer is unique. These facts follow immediately from Theorems 3.1 and 4.1 in Acar and Vogel (1994).*

4 FURTHER RESULTS AND OPEN ISSUES

Several other results on existence and uniqueness of global minimizers, under different conditions on θ and T can also be established. We state next, without proof, one of them.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a normed vector space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $v \in \mathcal{Y}$, α_1, α_2 positive constants and $\theta : \Omega \to [0,1]$ such that $\theta \in C^1(\Omega)$, $\frac{1}{1-\theta} \in L^1(\Omega)$ and $\frac{1}{\theta} \in L^\infty(\Omega)$. Then the functional (8) has a unique global minimizer $u^* \in BV(\Omega)$.*

It is important to mentioned at this point that, although numerical results are quite promising, we were unable to proof any rigorous mathematical results on existence for the important case corresponding to θ binary (i.e. with values in the set $\{0, 1\}$). We are still devoting efforts to this case. Also, in light of the results in Acar and Vogel (1994), we establish the following conjecture, which we were unable to prove up to now.

Conjecture 4.2 *Assume there exist a set $\Omega_1 \subset \Omega$ of positive measure such that $\theta(x) = 1 \ \forall x \in \Omega_1$. We conjecture that:*

(i) If $T\chi_{\Omega_1} \neq 0$ then J_θ has a global minimizer.

(ii) If moreover $T|_{\Omega_1}$ is injective, such a global minimizer is unique.

5 APPLICATIONS TO SIGNAL AND IMAGE RESTORATION

The purpose of this section is to present some applications of the simultaneous use of penalizers of L^2 and of bounded-variation (BV) type to signal and image restoration problems.

Example 5.1: A basic mathematical model for signal blurring is given by convolution via the following Fredholm integral equation of first kind:

$$g(t) = \int_0^1 h(t,s) f(t) ds, \tag{14}$$

where $h(t,s) = \frac{1}{\sqrt{2\pi}\sigma_b} \exp\left(-\frac{(t-s)^2}{2\sigma_b^2}\right)$ is a Gaussian kernel, $\sigma_b > 0$, f is the original signal and g is the convolved signal. For the numerical examples that follow, equation (14) was discretized in the usual way (using collocation and quadrature), resulting in a discrete model of the form

$$v = Au, \tag{15}$$

where A is a $n \times n$ matrix, $u, v \in \mathbb{R}^n$ ($u_j = f(t_j)$, $v_j = g(t_j)$, $t_j = \frac{j}{n}$, $1 \leq j \leq n$). For this case we considered $n = 350$ and $\sigma_b = 0.05$. The data v was contaminated with a 0.1 %

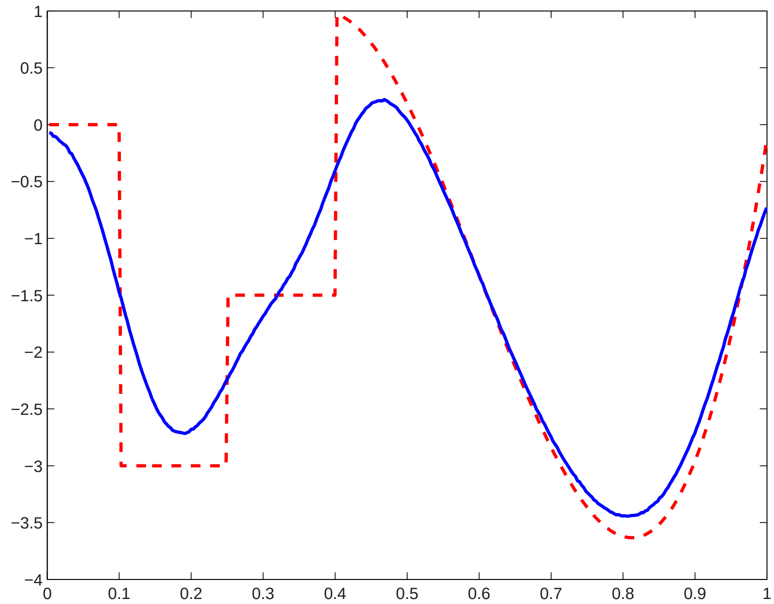

Figure 1: Original signal (- -) and convolved noisy signal (—).

zero-mean Gaussian additive noise (i.e. standard deviation of the order of 0.1 % of $\|v\|_\infty$). Figure 1 show the original signal (unknown in real life problems) and the blurred noisy signal which constitutes the data for the inverse problem.

Figure 2 show the regularized solutions obtained with the classical Tikhonov-Phillips method of order zero with regularization parameter $\alpha = 1 \times 10^{-6}$ and with penalizer associated to the bounded variation seminorm J_0 with regularization parameter $\alpha_{BV} = 0.1$ (in this case an algorithm proposed in Jensen et al. (2012) was utilized). Comparing the regularized solutions in Figure 2, it's clearly seen how the regularized solution obtained with the J_0 penalizer is significantly better then the one obtained with the classical Tikhonov-Phillips method near jumps and in regions where the exact solution is piece wise constant. Its also observe that the opposite happens where the exact solution is smooth.

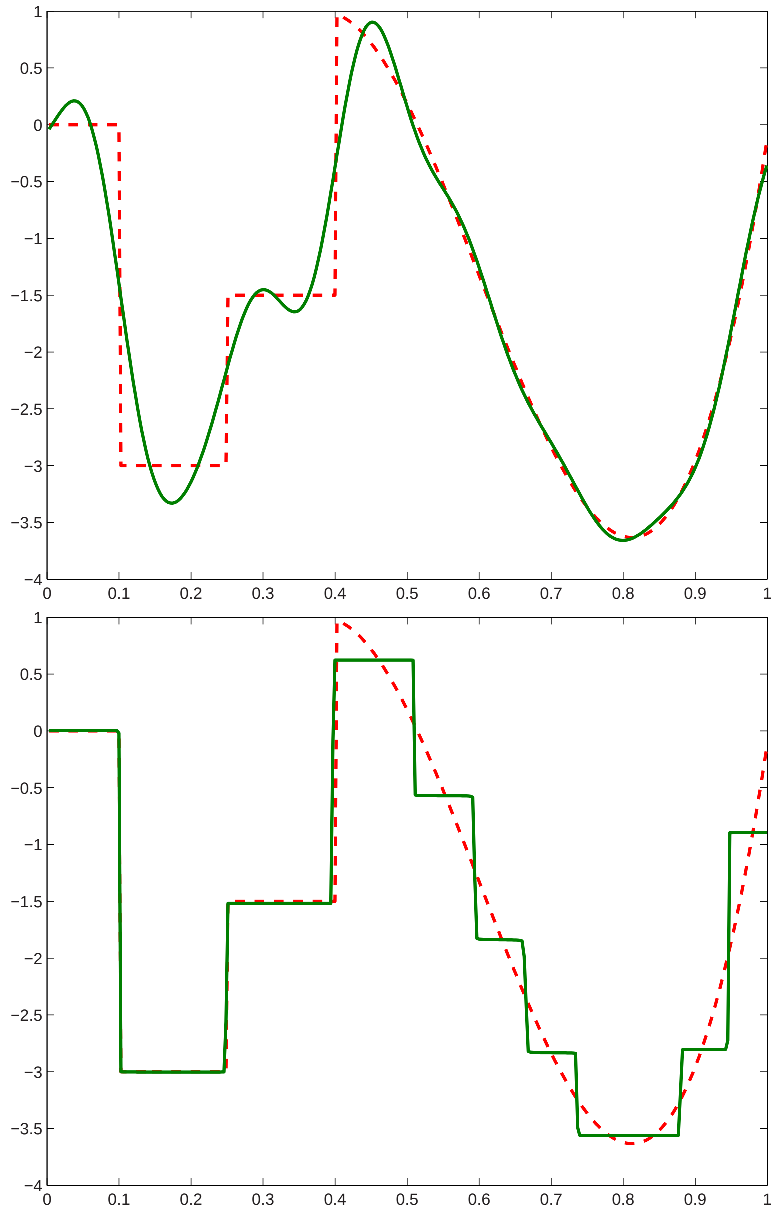

Figure 2: Regularized solutions obtained with Tikhonov-Phillips (top) and bounded variation seminorm (bottom).

Figure 3 shows the regularized solution obtained with the new combined $L^2 - BV$ (see (8)) with regularizations parameters $\alpha_1 = \alpha_2 = 1 \times 10^{-6}$. In this case the function $\theta(x)$ was chosen to be $\theta(x) \doteq 1$ for $x \in (0, 0.4]$ and $\theta(x) \doteq 0$ for $x \in (0.4, 1)$. Although this choice of $\theta(x)$ is clearly based upon "*a-prior*" information about the regularity of exact solution, other choices of θ can be made by using only data information. The improvement of the combined $L^2 - BV$ method with respect to both previous ones, is obvious. Nevertheless the improvement is also clearly reflected by the Improved Signal-to-Noise Ratio (ISNR) defined as

$$ISNR = 10 \log_{10} \left(\frac{\|v - u\|^2}{\|u_\alpha - u\|^2} \right),$$

(where u_α is the restored signal obtained with regularization parameter α). For the presented examples, the ISNR was computed in order to have an objective parameter to measure and compare the quality of all regularized solutions (see Figure 4).

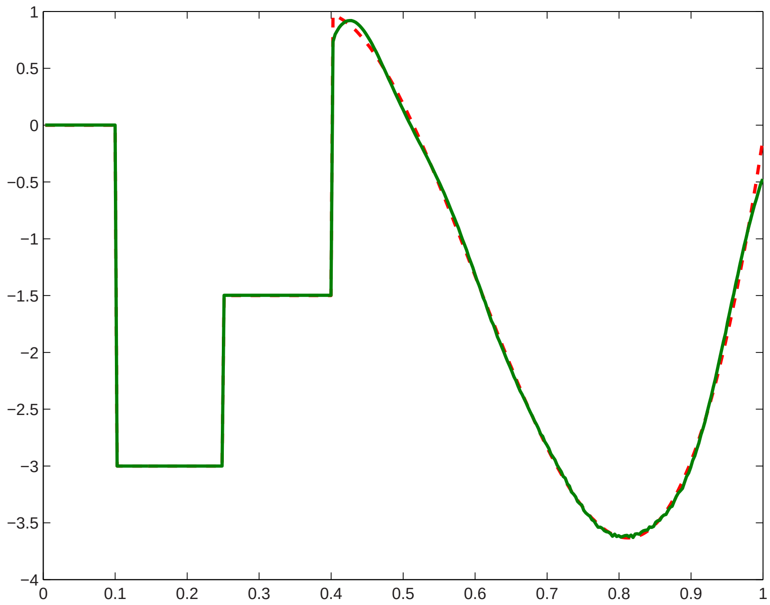

Figure 3: Regularized solution obtained with the combinated method L^2- BV.

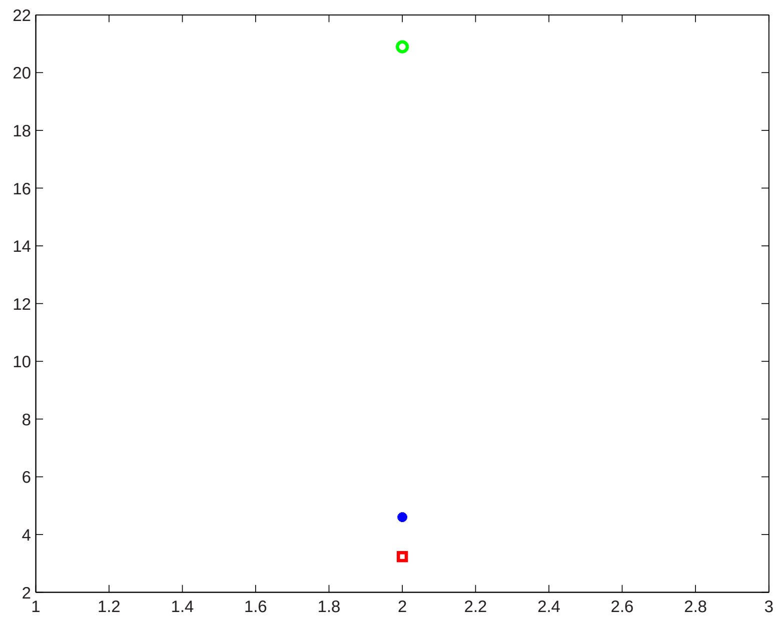

Figure 4: ISNR: Tikhonov-Phillips (red), bounded variation seminorm (blue) and combinated method L^2-BV (green).

Example 5.2: In this example we present an application to image restoration. For this case the forward blurring model is given by convolution with a point spread function of "atmospheric turbulence" type, i.e., with a two-dimensional Gaussian function with horizontal and vertical standard deviations σ_h and σ_v, respectively. Data for the inverse problem is then obtained by adding a $\sigma\,\%$ zero-mean gaussian noise to the blurred image. Figure 5 show the original and blurred noisy images corresponding to $\sigma_h = \sigma_v = 3$ and $\sigma = 0.001$.

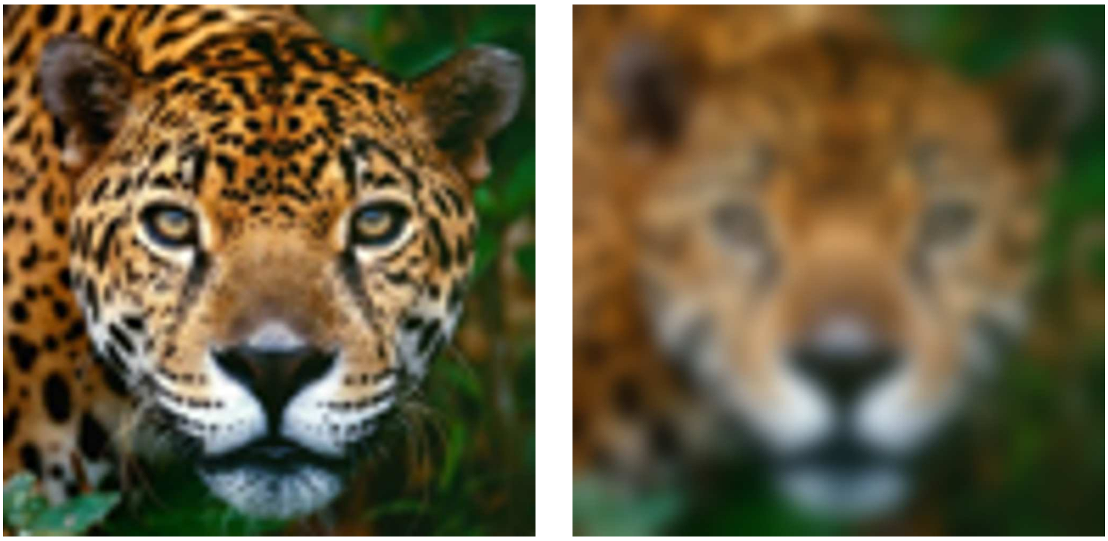

Figure 5: Original and blurred noisy images.

Figure 6 contains the regularized solutions obtained with Tikhonov-Phillips method and the use of penalizer associated to the bounded variation seminorm (in this case the algorithm proposed in Dahl et al. (2010) was utilized). Figure 7 shows the restoration image obtained with the new combined $L^2 - BV$ method. For this new method, regularization was numerically obtained as the minimizer of the functional (8) with $\theta(x)$ computed at each pixel, as a function of the modulus of the gradient of u (with an appropriate threshold). Finally, Figure 8 show the ISNR values corresponding to each method.

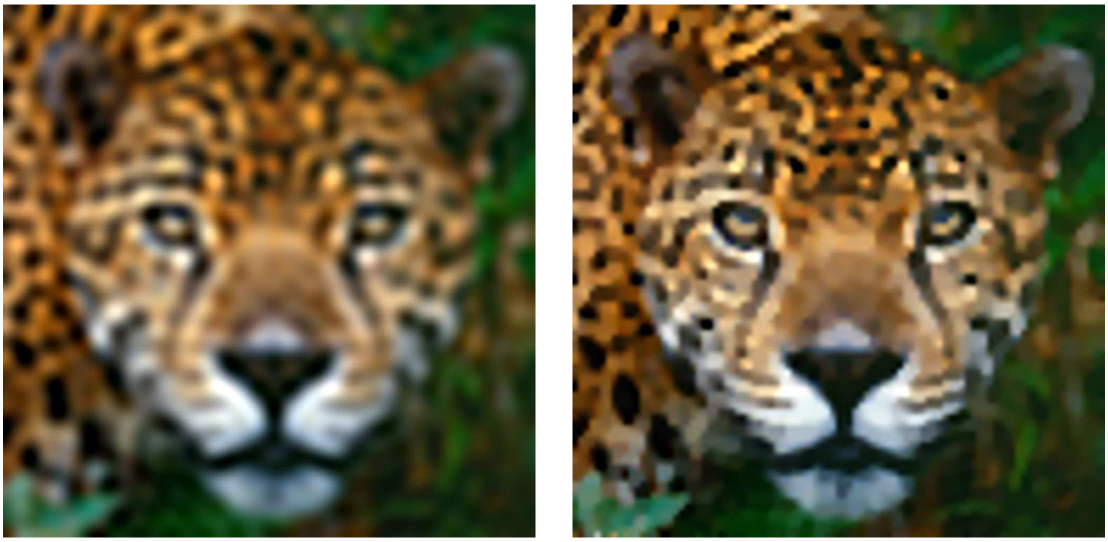

Figure 6: Regularized solutions obtained with Tikhonov-Phillips and bounded variation seminorm.

Figure 7: Regularized solution obtained with the combinated method L^2- BV.

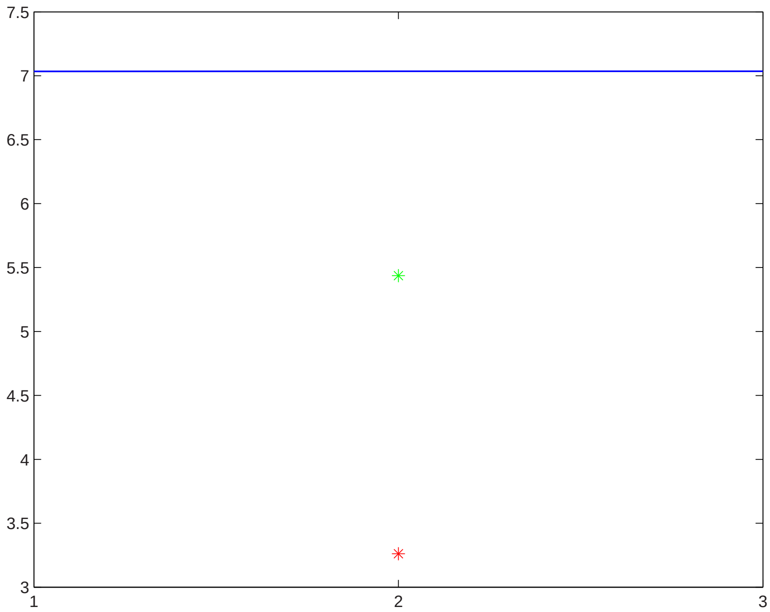

Figure 8: ISNR: Tikhonov-Phillips (red), bounded variation seminorm (green) and combinated method L^2-BV (blue).

6 CONCLUSIONS

In this work we presented some new results on the simultaneous use of penalizers of L^2 and of bounded-variation (BV) type. For particular cases, existence and uniqueness results were shown. Open problems were discussed and several results in applications to signal and image restoration problems were presented. Although these preliminary results are clearly quite promising, much further research is needed. In particular, besides Conjecture 4.2, and in spite of interesting numerical results, no rigorous mathematical proofs are yet known on the existence and uniqueness of minimizers of functional (8) for the case $\theta(\cdot)$ binary (i.e. with $\theta(x)$ taking only the values 0 and 1).

Another important research direction is in regard to the “optimal” choice of the parameters α_1 and α_2. It is not clear how, if in any way, those parameters are related to the corresponding optimal values (as given for instance by the $L-$curve method or by Morozov’s discrepancy principle) of the pure simple cases, Tikhonov of zero order and BV-regularization. The choice of $\theta(\cdot)$ in a somewhat optimal way is also a subject which deserves much further attention. Research in all these directions is currently under way.

ACKNOWLEDGEMENTS

This work was supported in part by Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET, through PIP 2010-2012 Nro. 0219, by Universidad Nacional del Litoral, UNL, through project CAI+D 2009-PI-62-315, by Agencia Nacional de Promoción Científica y Tecnológica, ANPCyT, through project PICT 2008-1301 and by the Air Force Office of Scientific Research, AFOSR, USA, through Grant FA9550-10-1-0018.

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