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# A COMPARATIVE ASSESSMENT OF METHODOLOGIES BASED ON MODAL ANALYSIS AND B-SPLINE INTERPOLATION FOR THE ESTIMATION OF STRESSES IN BEAMS

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**Abstract**. In this work two methodologies for the estimation of stresses on a cantilever beam by sequentially measuring accelerations in nine different locations are compared. The first procedure uses modal analysis, specifically the experimentally determined mode shapes, to predict the stress spectrum and the time response to an impulse excitation based on the signals from the accelerometers. The second methodology is based on B-Spline approximation of the displacement fields method using Least Squares Error Minimization. In both methodologies numerical integration is required to obtain the displacement from the acceleration field. Three different integration techniques, conducted in time and frequency, respectively have been implemented and compared. The signals estimated with both methods are compared with the experimental stresses, measured with a strain gauge placed on the structure. A good prediction of the both the stress spectrum obtained under white noise excitation and the impulse response time signal is obtained with both methods, with the B-Spline method offering a simpler and more straightforward approach. In either case the possibility of reliability measuring stress in a cantilever beam using accelerometers is successfully demonstrated.

# 1 INTRODUCTION

To ensure the health of a structure traditional engineering methods rely on safety factors applied during the design phase, accounting for both the lack of information regarding materials properties, the geometry of the structure actually built, and uncertainties in the engineering models used (Budyas & Nisbett, 2008). Structural health monitoring (Balageas et al., 2010), on the other hand, allows to obtain information about the stress state of the structure, ideally on a continuous basis and with high time resolution, in order to predict damage caused by extreme instantaneous stresses or fatigue stresses. Continuous stress monitoring ideally allows for a reduction of safety factors, leading to lighter and more economical structures, and estimate accumulated damage from fatigue, while simultaneously being capable of issuing warnings if potentially critical conditions are reached. Stresses are typically measured with strain gauges which are highly accurate and reliable under laboratory settings but are often not practical under ambient conditions due to problems with deterioration and non-trivial instrumentation.

While strain gauges are a very direct way of assessing stresses, indirect methods may be just as accurate and potentially more robust, easier to implement, and with increased flexibility. One such method is based on the use of compact accelerometers (nowadays widely available because of the large market for smart phones and other smart devices) which can be placed on a structure of interest and continously monitored using modern information technology. As demostrated by Pelayo et al. (2015) and Aenlle et al. (2017), modal analysis can be used to accurately reconstruct stress signals, both in the frequency and the time domain. Their method relies on analytical expressions for the relation between the stress at a given location and for a given direction and second derivatives of the displacements, demonstrated for the case of simple structures such as cantilever beams and flat plates. In order to infer the stress signal at a given location the mode shapes of the structure have to be determined, either numerically, through a dedicated laboratory experiment, or through operational modal analysis.

The objective of the present work was to assess if the same instrumentation used by the modal analysis approach could be utilized to obtain the stress signals at an arbitrary fixed location in a more straightforward way, specifically using Bspline interpolation of the displacement field, allowing to obtain the required derivatives in a direct way while making a similar use of the redundancy conferred by the multiple-point measurement. The rest of the paper is organized as follows: in the methodology section 2 we first review the theoretical background for stress determination using modal analysis for the case of a simple Euler beam, briefly describe the integration techniques used to obtain the displacement field from the acceleration field, and finally formulate the simple Bspline interpolation technique proposed as an alternative method for obtaining the stress signal. In section 3 results obtained with a nine-point modal analysis setup are presented. First, the effect of using three integration techniques on the stress signal is discussed. We then go on to show the results obtained with an implementation of the modal analysis technique for stress recovery and their validation against a strain gauge signal. The final section deals with the innovation of the present work, the demonstration that a B-Spline interpolation technique allows for a stress signal reconstruction just as accurate as the somewhat more complex modal analysis technique. Section 4 summarizes the results and presents some conclusions.

# 2 METHODOLOGY

## 2.1 Method 1: Stress estimation through modal analysis

For a cantilever beam, the Euler Bernoulli theory is assumed (see Figure 1). The bending moment and the curvature are then related by:

$$EI_y \frac{\mathrm{d}^2 u}{\mathrm{d}^2 x} = M_y \tag{1}$$

Where $E$ is the Young's modulus, $I_y$ is the second moment of the cross section about $y$ axis, $u$ is the vertical displacement, and $M_y$ is the bending moment. The interest of this work is the obtaining of stresses. The stresses can be determined (at any section located at the distance $x$) with Navier's law

$$\sigma(x) = \frac{M_y}{I_y} z\,, \tag{2}$$

where $z$ is the distance from the neutral axis to the point of interest in the section. Combining (1) and (2) the following relation between stress and curvature is obtained:

$$\sigma(x) = -Ez \frac{\mathrm{d}^2 u}{\mathrm{d}^2 x} \tag{3}$$

Given the discrete nature of the measurement array a finite-element formulation is useful. In this formulation the displacement at an arbitrary point in the beam element (see Figure 2) can be obtained as (Hutton, 2003):

$$u(x) = \{N^e(x)\}\{u^e\}\;, \tag{4}$$

where $\{N^e(x)\}$ and $\{u^e\}$ are vectors containing the element shape functions (Hutton, 2003) and the nodal displacements corresponding to the element $e$ respectively. Calculating the second derivate of the shape functions with respect to $x$, and introducing the time dependency in the displacement vector in, the following expression is obtained:

$$\sigma(x,t) = -E\{N^{e''}(x)\}\{u^e(t)\}z \tag{5}$$

The vector $\{u^e\}$ can be expressed in terms of the mode shapes of the structure $[\phi^e]$, by means of the modal coordinates (Craig and Kurdila, 2006):

$$\{u^e(t)\} = [\phi^e]\{q(t)\} \tag{6}$$

Inserting equation (6) into (5), the stress at an arbitrary location $x$ along the beam, located in the beam element $e$ can be calculated from:

$$\sigma(x,t) = -E\{N^{e''}(x)\}\{[\phi^e]\{q(t)\}z \tag{7}$$

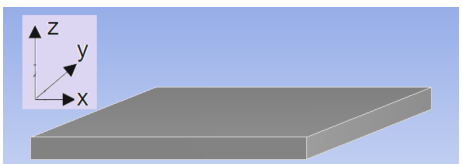


**Figure 1**. Euler Bernoulli beam

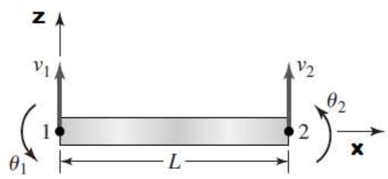


**Figure 2.** Beam element nodal displacements.

### Experimental mode shapes of the structure

The mode shapes of the structure have been identified experimentally. A simple way of doing so is the use of a technique called *Quadrature Picking*, based on the following reasoning. At any frequency, the magnitude of the Frequency Response Function (FRF) is the sum of the contributions from all modes at that particular frequency. For Single Degree of Freedom Systems (SDFS), the frequency response function at resonances is purely imaginary. As a result, the value of the imaginary part of the FRF at a resonance (for structures with lightly coupled modes) is proportional to the modal displacement (Gade et al, 2005). Since the condition of light coupling was well observed in the present experiment this procedure was used in this work. Measurements were taken sequentially in nine equidistant points and the magnitude of the imaginary part of the FRF was recorded for each resonance. In a final step the modal displacements were mass-normalized for the construction of the mode shape matrix $[\phi^e]$.

### Modal coordinates

Using equation (6), with the mode shapes of the experiment and with the responses of the accelerations (measured in 9 points of the structure), the modal acceleration coordinates are obtained by:

$$\{\ddot{q(t)}\}_{exp} = [\phi]^{+}{}_{exp}\{\ddot{u}\}_{exp} \tag{8}$$

Where $\{\ddot{u}\}_{exp}$ are the accelerations measured. In order to obtain the modal displacement, a double integration of equation (8) is required, for which the time-domain integration techniques presented in Brand (2011) and Pintelon & Schoukens (1990), and the frequency-domain integration technique proposed by Mercer (2006) are used. In all cases the modal acceleration signals are first filtered through a high-pass filter to avoid integration errors produced by low frequencies. The *Cumtrapz* command from MATLAB is then used for integration in time. The second-time integration method uses a fifth-order IIR filter presented (Pintelon and Schoukens., 1990). The frequency integration, finally, is performed with the Omega Arithmetic (OA) method (Mercer, 2006),

## 2.2 Method 2: Stress estimation through B-spline Interpolation

B-splines are a set of Bezier functions in a piecewise configuration. These types of functions are useful for the construction of smooth curves, surfaces and volumes, ensuring $C^K$ continuity of the geometry when differentiations are required. B-spline functions can be denoted as $N_{i,p}(\xi)$, where $p$ is the order of degree of the Bezier functions, $i$ es the $i$-th Bezier function and $\xi$ is the parameter space vector where the curve is mapped. Non-Uniform Rational B-spline (NURBS) are the general description of B-splines, where the knot vector sequence can be non-uniform and each control point or basis function has a specific weight $w_i$ (Cottrell et al., 2009)

$$R_{i,p}(\xi) = \frac{N_{i,p}(\xi)w_i}{\sum_{i=1}^{n} N_{i,p}(\xi)w_i} \tag{9}$$

A NURBS curve Q(ξ) is constructed using the NURBS functions R(ξ) as:

$$Q(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\, P_i \tag{10}$$

where n is the number of control points $P_i$. In the present case $Q(\xi) = u(x)$, where $u$ is the displacement at the beam coordinate $x$ as before. Equation (3) for the stress can then be conveniently cast into the form

$$\sigma(x) = Ez \sum_{i=1}^{n} \frac{d^2 R_{i,p}(x)}{dx^2} P_i \tag{11}$$

Piegl (1996) presents several algorithms for the calculation of derivatives of B-spline functions. One important requirement in curve and surface generation methods is the ability to construct a geometry with only a few control points for a further use in numerical analysis. In this work a Least Squares Minimization (LSM) method was used (Chen et al. 2010).

### 2.3 Experimental setup

For the experiment, a cantilever beam of aluminum was used with the following measurements: height = 3mm, width = 37.83 mm, and length = 843 mm. At 50 mm from the free end, the beam was excited by means of a programmable shaker. On the beam, accelerations measurements were made sequentially at 9 different points (see Figure 3). A Brüel & Kjær commercial modal analysis system equipped with accelerometers 4535-B-001 and the RT Pro Photon software was used; the sampling frequency was 16394 Hz at all times.

All three Cartesian components were acquired but only the vertical component was used in the analyses, given the simple geometry of the setup. White noise excitation was used to obtain the accelerometer spectra. For time-domain analyses the shaker was programmed to generate periodic impulses and the complete time series for a 1-second period was collected.

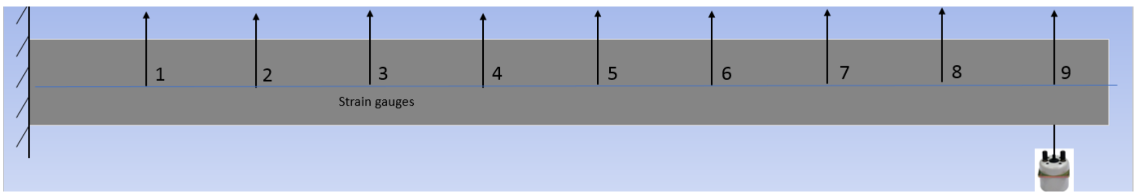


**Figure 3.** Experimental setup

The two 120 Ω strain gauges were instrumented with a home-built half bridge arrangement and conditioning circuit, and the output signal was acquired with a National Instruments data NI 6009 data acquisition board connected to the LabVIEW data acquisition software. Accelerometer and stress signals were aligned manually taking advantage of the impulse excitation which marks a clearly identifiable start point of the time series.

## 3 RESULTS AND DISCUSSION

### 3.1 Experimental mode shapes

Figure 4 presents the experimental results obtained for the imaginary part of the FRF for the first four modes at the 9 different points of the experiment, obtained with a white noise shaker excitation. The shapes of the modes can be recognized by simple visual inspection. Each spectrum corresponds to one only measurement run; no attempt was made to increase the quality of the spectrum by averaging or filtering given the good quality of the raw results.

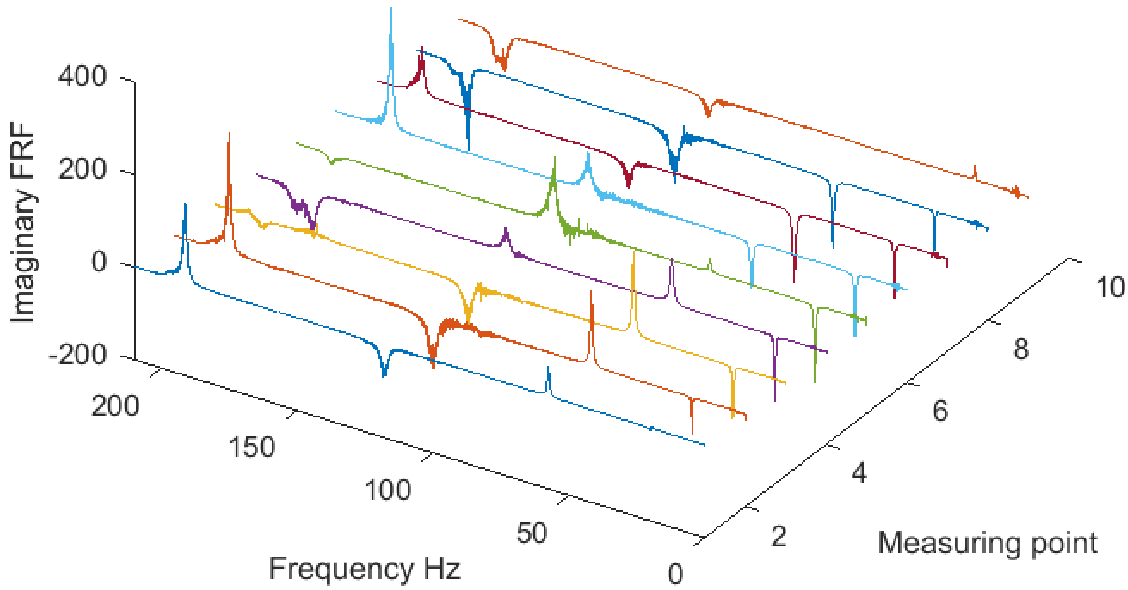


**Figure 4.** Imaginary part of FRF at the 9 points of the experiment

As explained in section 2.1 the mode shapes of the experiment can be obtained from the imaginary part of the FRF. Afterwards, they are mass-normalized and compared with the corresponding results of finite-element (FE) analysis conducted with Ansys. A comparison of the experimental and FE mass-normalized mode shapes are shown in Figure 5; it can be seen that a fair agreement between the experimental and the FE results has been reached.

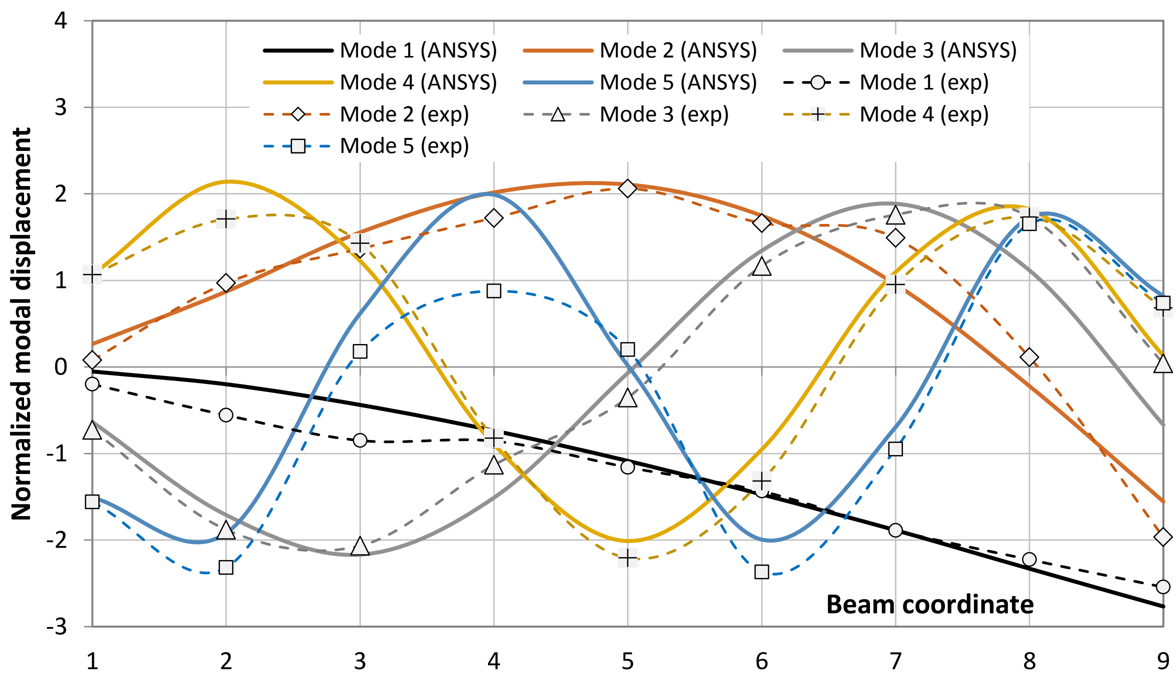


**Figure 5**. Experimental vs simulated (FEM model) mode shapes for the first five beam modes

## 3.2 Stress estimation through modal analysis.

As a first demonstration of the stress signal reconstruction using modal analysis the impulse response was calculated. In this and all subsequent cases 40 realizations of the impulse response were averaged into one time series. Figure 6 shows the predicted stress signal at the strain gauge position calculated with the equation (7), where the experimental mode shapes shown in Figure 5 were used. Modal accelerations were calculated with equation (8), and the modal displacements were obtained with the three integration techniques mentioned in section 2.1. As seen in Figure 6, the three integration methods produce practical identical stress spectra with a predominant peak at 54 Hz and very similar time series.

The stresses obtained with the strain gauge measurements (placed at 25 cm from the fixed end) are presented in Figure 7. The signal shown stems from the averaging of 40 signals acquired sequentially with a train of impulses imparted by the shaking device. The measured signal can be seen to be very similar to the one reconstructed from the accelerometer measurements, except for a very low-frequency component which appears to be reflect a drift component or other slow laboratory noise which could be easily eliminated by high-pass filerting. All resonances seen in the reconstructed signal (around 20Hz, 54Hz, 105Hz, and 125Hz) are also seen in the measured stress signal, where the 20Hz component was found to be somewhat stronger than predicted from the accelerometer readings. It should, however, be mentioned that accelerometer readings were acquired sequentially, requiring relocations of the accelerometer between runs, which introduces additional uncertainty given the errors involved with the experimental handling and positioning process. In future work a custom-made accelerometer array will allow for a simultaneous acquisition of all accelerometer signals.

To facilitate the comparison between the estimated and the predicted stress time series the results obtained with two of the prediction methods (time integration with IIR filters, frequency integration with the OA method) have been plotted into a single graph with the experimental (strain gauge) time series. These three curves are shown in Figure 8. It can be seen that the experimental stress signal is quite accurately predicted, with the main difference being a somewhat lower damping of the 20Hz component (as evidenced by the higher peak height to width ratio of the corresponding resonance in the experimental stress spectrum).

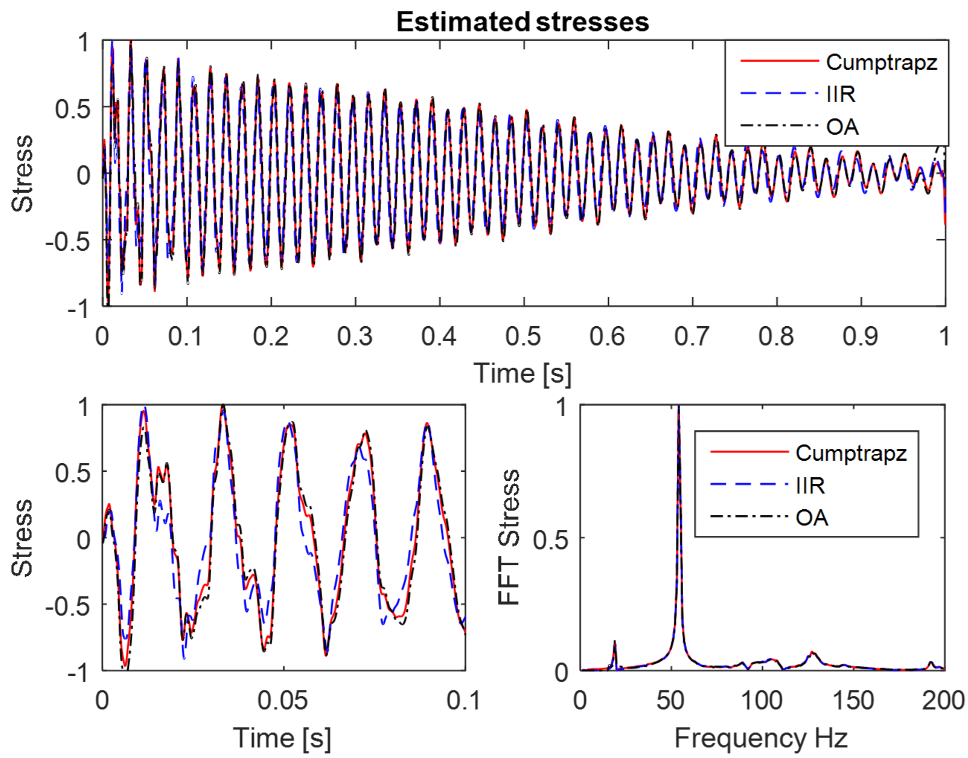


**Figure 6.** Estimated stresses using modal analysis and different integration methods

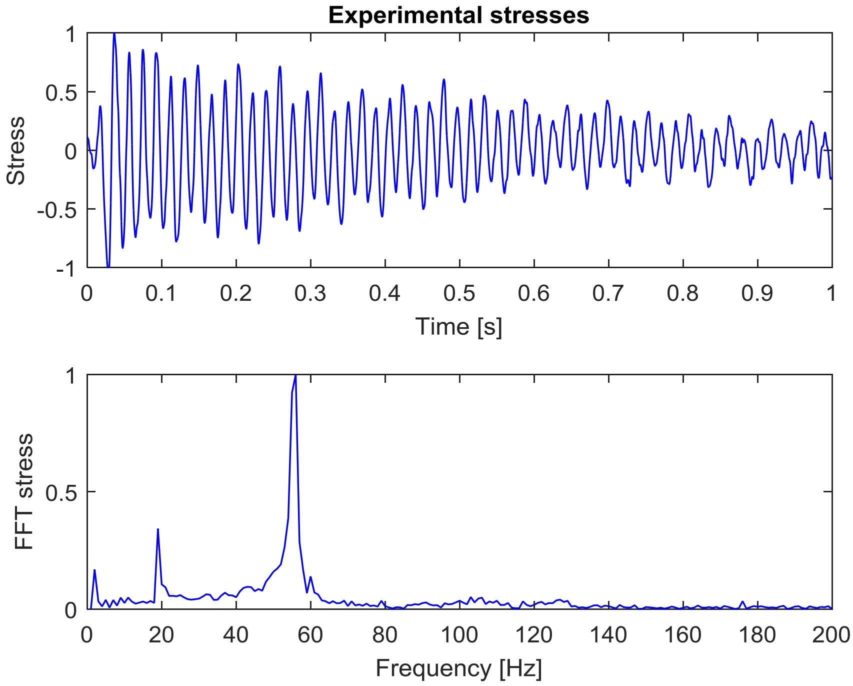


**Figure 7**. Experimental stresses as measured with the strain gauge

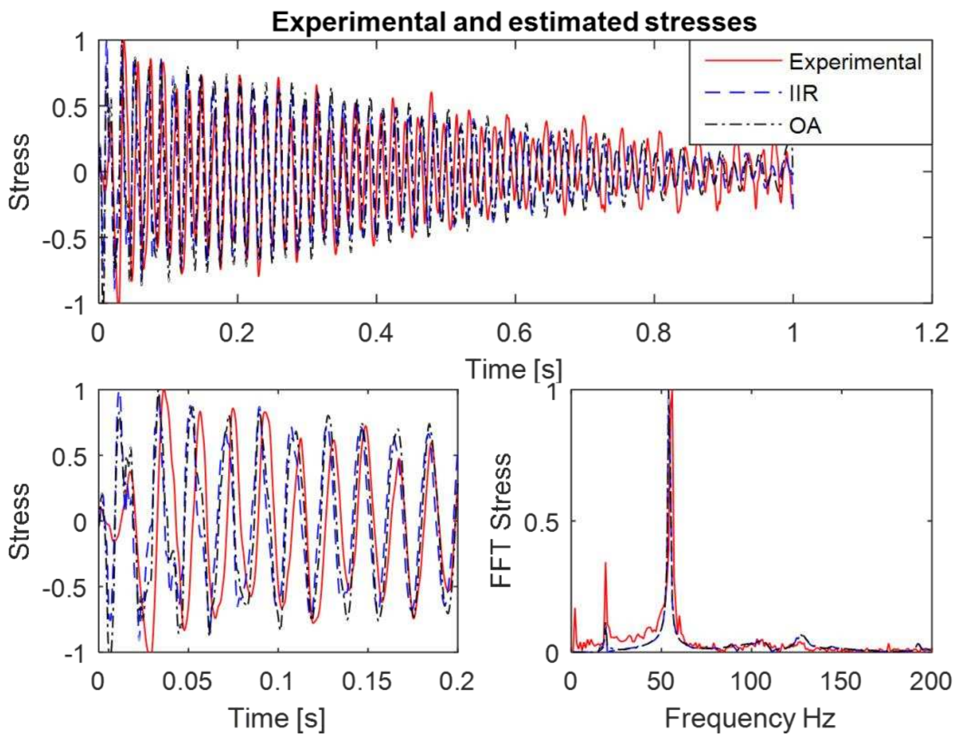


**Figure 8.** Experimental and estimated stresses

## 3.3 Stress estimation through B-spline interpolation

In the following the results obtained with the second method discussed in the present paper will be shown. Equation (11) was used to calculate the stress at the strain gauge position $x =$

$x_{SG}$ by calculating the second derivatives of the B-Spline interpolation functions $R_{i,p}(x)$ for the displacement field $u(x)$ at $x_{SG}$. As before, numerical integration was used to calculate the displacement field from the accelerometer field. As shown in Figure 9 the FFT stress spectrum predicted by this method is very similar to the one obtained with the modal analysis approach, with a slightly left-shifted main peak (by about 2 Hz) and slightly higher presence of the higher modes at 105 Hz and 125 Hz. As before, the 20 Hz peak is lower (at a similar absolute width), leading to a higher attenuation of this component in the predicted compared to the measured signal. This discrepancy appears to have nothing to do with the processing of the displacement field and is likely to be a consequence of the experimental errors related to the repositioning of the accelerometer between measurements, as mentioned above.

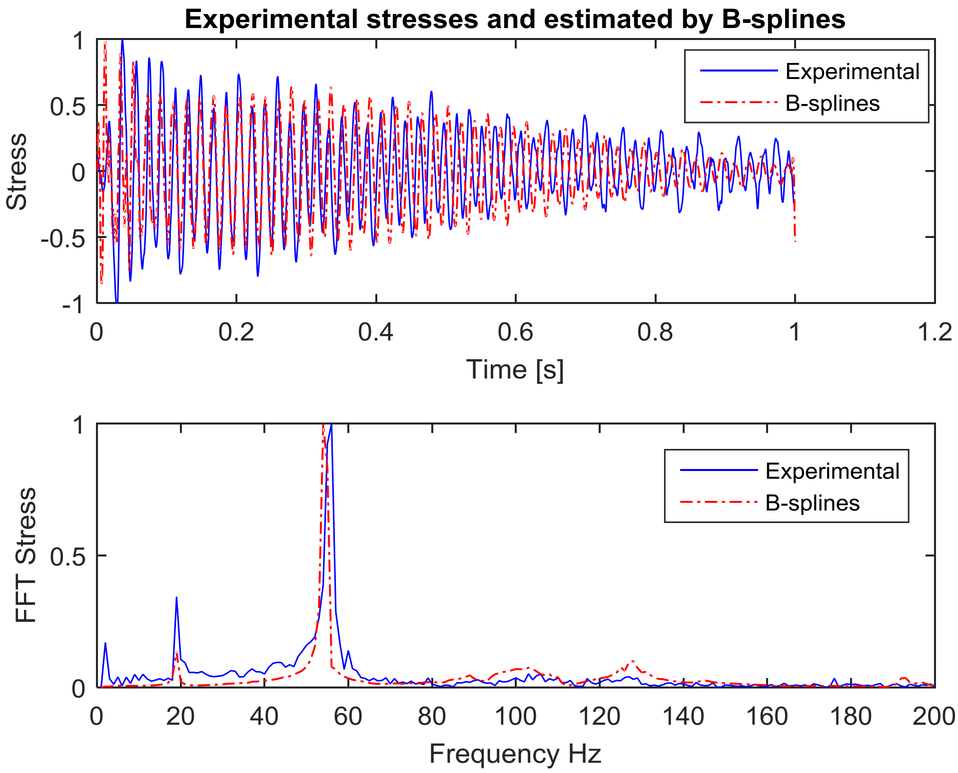


**Figure 9**. Estimated stresses by B-splines

## 4 SUMMARY AND CONCLUSIONS

A comparative assessment between a modal-analysis and a B-Spline-interpolation technique for the estimation of stresses at an arbitrary point of a cantilever beam has been presented. Method 1 combines modal analysis and a finite-element formulation to relate the point stresses with the experimentally determined mode shapes and the experimental displacement field time series. The displacement signals are obtained through integration of the accelerometer time series. Nine equidistant accelerometer positions were used in the present work, where the accelerometer is repositioned between measurements.

Three different numerical integration methods (combined with digital filters) have been used, including two-time integration methods (trapezoidal method and IIR filters) and a frequency integration method (Omega Arithmetic), and have been shown to lead to very similar (though not identical) results.

The comparison between the predicted stress spectra and time series with the corresponding results obtained separately in a two strain gauge arrangement shows a good agreement with the modal analysis-based approach and the strain gauge measurements, as expected from the inspection of similar results reported in literature. An additional method based on B-Spline fitting of the displacement field (obtained through numerical integration, as in the first method), while being conceptually more simple, was found to produce similar results as the modal analysis method. The slight discrepancies observed upon comparing with the strain gauge results are very similar to the ones obtained with the modal analysis approach and are attributed to the experimental uncertainties associated with the repositioning of the accelerometers between measurements. Future efforts will use a custom-made accelerometer array with MEMS-type sensors, allowing for a simultaneous acquisition of all accelerometer signals.

## REFERENCES


Aenlle, M., Pelayo, F., and Canteli, A., Fatigue damage detection and prediction of fatigue life on a cantilever beam. *International Journal of Structural Integrity,* https://doi.org/10.1108/IJSI-01-2017-0001.

Balageas, D. (ed.), Claus-Peter Fritzen, C.-P. (ed.), Güemes, A. (ed.), Structural Health Monitoring, John Wiley & Sons, 2010

Budyas, G.B., Nisbett, J.K. Shiqley's Mechanical Engineering Design, 8$^{th}$ edition, McGraw-Hill, 2008

Brandt, A., Noise and vibration analysis: signal analysis and experimental procedures. *Willey*, United Kingdom, 2011.

Chen, X., Ma, W., and Cubic, J., “Cubic B-spline curve approximation by curve unclamping,” *Comput. Aided Des.*, vol. 42, no. 6, pp. 523–534, 2010.

Cottrell, J., Hughes, J., and Bazilevs, Y., *NURBS as a Pre-Analysis Tool: Geometric Design and Mesh Generation*. 2009.

Craig, R., and Kurdila, A., Fundamentals of structural dynamics, 2nd edition, 2006.

Gade, S., Herlufsen, H., and Konstantin, H., How to Determine the Modal Parameters of Simple Structures, Application Note, *Brüel&Kjær,* Denmark, 2005.

Hutton, D., Fundamentas of Finite Element Analysis. *McGraw-Hill,* 2003.

Mercer, C., Acceleration,Velocity and Displacement Spectra. *Omega Arithmetic*, November 2006.

Pelayo, F., Skafte, A., Aenlle, M., and Brincker, R., Modal Analysis Based Stress Estimation for Structural Elements subjected to Operational Dynamics Loadings. *Experimental Mechanics*, 9(55), 1781-1802, 2015.

Piegl, T., *The NURBS Book*, vol. 28, no. 8. 1996.

Pintelon, R., and Schoukens, J., Real-time integration and differentiation of analog signals by means of digital filtering, *IEEE Transactions on Instrumentation and Measurement,* vol. 39, no. 6, pp. 923-927, Dec. 1990. doi: 10.1109/19.65799.