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BAYESIAN INFERENCE OF CONSTITUTIVE PARAMETERS THROUGH THE COLLISION DYNAMICS OF A BALL

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Abstract. The trajectory of a ball impacting with an angle on a rigid boundary is recorded with a high-speed camera and the dynamics is reconstructed in a computer. Several experiments are carried out in order to obtain statistical distributions of the trajectory. On the other hand, a simple model of a viscoelastic material ball simulates the experiment. If the values of the constitutive parameters (*e.g.* elastic and viscous modulus, friction coefficient, etc.) in the numerical model are correct, the simulated dynamics and the experimental data must be comparable. In this study, the Bayesian inference is applied to identify two constitutive parameters (the friction and viscous coefficients) through statistical measures. Furthermore and once the distributions of the stochastic parameters are available, a propagation of uncertainties is carried out to verify the experimental results and also, to obtain information about other variables of interest which experimental determination is not straightforward, such as contact forces, time of contact, etc.

1 INTRODUCTION

The computational modeling of a material behavior is based on equations which represent the response of the material to forces, temperature, strains, etc. The constitutive equations involve parameters that vary from a specific material to other and they are usually estimated through the use of experimental tests, under assumptions that simplify the estimation (*e.g.* low strain rate, plain stress, plain strain, etc.). In particular, viscoelastic materials show a strain-rate dependent behavior and several models and parameters to represent them are available in the literature. As is known, there are also many different tests used to determine these viscoelastic parameters (Larson, 1999). On the other hand, linear viscoelastic, surface friction and plasticity theories have contributed to address deformations and stresses when inelastic bodies are in contact. The works as Popov (2010); Johnson (1987) can be cited among the books including modern approaches in this field. In more specific cases dealing with contact between very hard and soft bodies, a discussion of how the resulting friction force depends on the nature of the substrate surface roughness and on the sliding velocity can be seen in Nikoofard et al. (2014); Palasantzas (2003). When the substrate surface has a self-affine fractal structure, it can be found in the comprehensive work of Persson (Persson, 2001).

There are different strategies to solve an inverse problem in order to estimate parameters, for example the least squares method, as in Vuoristo et al. (2000) where the algorithm of Levenberg-Marquadt was used to adjust data from a dynamic compression test to a spring-dashpot model for viscoelasticity. Sometimes, when the inverse problem is ill-conditioned, the use of gradient search methods is not always useful to solve the problem. Heuristic techniques such as neural networks and genetic algorithms are successfully employed to overcome the difficulties. For instance, the crack detection in structural elements can be addressed using neutral networks (Rosales et al., 2009) and genetic algorithms (Buezas et al., 2011). Bayesian inference (Kaipio and Somersalo, 2006) is traditionally used for model selection, as in Beck and Yuen (2004); Toni et al. (2009); Ritto and Nunes (2015), but it can also useful for parameter estimation (Toni et al., 2009; Ritto, 2014).

In the present paper, the problem of a viscoelastic ball impacting on a rigid body is studied and the Bayesian framework is implemented to determine the viscoelastic and friction parameters. Based on the outcomes of an elementary experimental setup, the statistics are input as data in the numerical representation of a non-linear mechanical model. The objective of this paper is to answer the question: Is it possible to measure constitutive parameters indirectly using a Bayesian approach? In order to respond to this question, a simple model (as simple as possible) of the ball is proposed. This model is stated in terms of ordinary differential equations which can be integrated numerically with a smaller amount of CPU time than the expected in solving a continuum mechanical scheme.

This paper is organized as follows. Section 2 presents the simple nonlinear mechanical model and its numerical implementation, the contact constraint and the friction effect of the oblique collision of the body. In Section 3, the experimental setup for measuring the kinematics of a real ball impacting on a rigid wall is described. Next, Section 4 depicts the identification procedure based on the Bayes theorem. Then, numerical results of the Bayesian procedure and the estimation of other dynamic quantities using the adjusted model are discussed in Section 5. Finally, the concluding remarks are included in Section 6.

Figure 1: Scheme of the bouncing ball and the interaction forces.

2 MECHANICAL MODEL

The physical approach is straightforward. The Newton's equations of a viscoelastic ball, bouncing over a rigid plane are stated and solved.

The interaction force between the sphere and the plane are modeled with two terms, *i.e.* the elastic part using the Hertz results and a term of viscosity. As the ball touches the rigid boundary, a dry frictional Coulomb provokes a spin. The motion involves three degrees of freedom. A scheme of the bouncing ball problem is shown in Figure 1.

The equations of motion when the ball hits the rigid are:

$$F_f = m\ddot{x}; \quad F_C + F_v = m\ddot{y}; \quad T = I\ddot{\alpha}$$

where m is the mass and r is the radius of the sphere. The interaction force (elastic part, Hertz model) is

$$F_C = \begin{cases} -\frac{4}{3}\sqrt{r}\frac{E}{1-\nu^2}y^{\frac{3}{2}} & if \quad y \leq 0 \\ 0 & if \quad y > 0 \end{cases} . \tag{1}$$

where E is the Modulus of Elasticity, ν is the Poisson ratio (Lifshitz et al., 1962), $F_f = -\mu F_C(\dot{x} + r\dot{\alpha})$ is the friction force (at the contact point), x and y are the coordinates of the ball (center of mass), $\ddot{x}$ and $\ddot{y}$ are the acceleration components, I is the moment of inertia of the sphere, $T = \mathbf{r} \times \mathbf{F} = rF_C$ is the moment due by friction force, $F_v = -\eta\dot{y}$ is the viscous force (acting only when the ball is in contact) and $\ddot{\alpha}$ is the angular acceleration. Therefore, the friction force F_f, the viscous force F_v and the torque T depend on the contact force F_C. The body is a rigid sphere that interacts with a rigid plane with a nonlinear, viscoelastic, Kelvin–Voigt material-like force. The equations of motion (1) together with the initial conditions constitute a nonlinear system of Ordinary Differential Equations (ODEs) that can be integrated numerically. Here, a Runge-Kutta method (Lasagni, 1988) is employed.

3 PHYSICAL EXPERIMENT

Several experiments were carried out to determine the probability distributions of the dynamics. The tests consisted in making a commercial polybutadiene ball with diameter of 25,6 mm impact against an inclined plane of hard wood (considered as a rigid body for modeling purposes). The experimental setup consists in two inclined planes. The upper plane includes a track to ensure that the ball always falls from the same starting point and following the same path. The lower board is placed so that the plane of rotation of the impacting ball is perpendicular to the plane of motion before and after the impact.

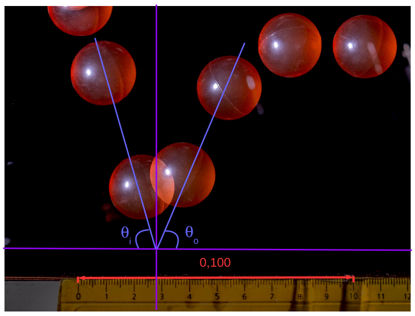


Figure 2: Superposition of frames extracted from the Tracker Software (Brown, 2015). The axes origin is the ball center of mass.

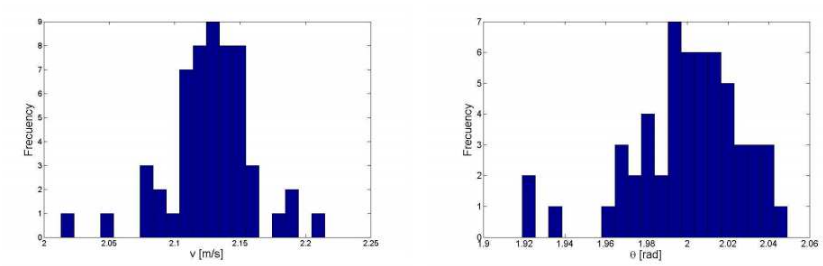
A total of 55 impact tests were performed and each one was recorded using a high-speed camera Casio Exilim ex-zr100 at 1000 frames per second. The camera was located in front of the impact plane in order to avoid the recorded trajectory to be affected by the rotation of the ball. The resulting videos were analyzed using the software Tracker (Brown, 2015). This program allows to measure distances after the input of a reference length along the horizontal axis. Thus, once the object is referenced, the automatic tracking of its motion is possible (see Figure 2). The coordinate points of the tracked object are registered for each time and the different characteristics of the trajectory can be measured. In particular, the variables of interest in the present study are the incidence angle (θ_i) and speed (v_i) and the deflection angle (θ_f) and speed (v_f).

The results of this series of 55 experiments were used to generate histograms for the rebound data (the incidence data were kept approximately constant since the height of the starting point and the angle of the inclined plane did not change). The obtained histograms are depicted in Figure 3a. Using these histograms, it is possible to estimate the probability density functions (PDFs) for both variables using MATLAB's *ksdensity* function (MATLAB, 2010; Scrucca, 2001). The resulting PDFs are shown in Figure 3b. The correlation between the variables was calculated using the Pearson correlation coefficient (Benesty et al., 2009), obtaining a value near to zero which means that a joint probability function $\rho_{exp}(v, \theta) = \rho_1(v) \cdot \rho_2(\theta)$ can be assumed.

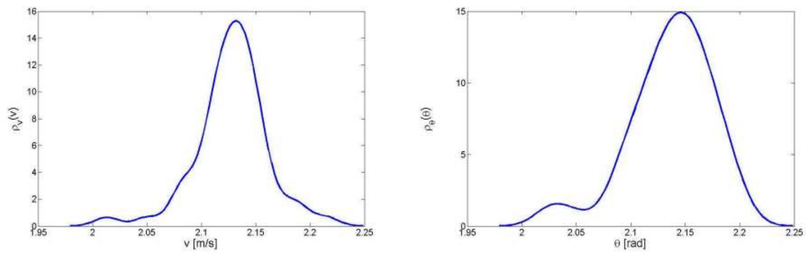
4 INFERENCE OF THE CONSTITUTIVE PARAMETERS USING A BAYESIAN APPROACH

Bayesian methods offer the opportunity to integrate data obtained from experiments and prior knowledge in order to derive an estimate of a posterior probability distribution of certain variables. In this work and in order to identify the two constitutive parameters $\mathbf{\Theta} = (\mu, \eta)$ in Eqs.(1,1), we propose the use of a Bayesian inference strategy (Kaipio and Somersalo, 2006) in which some parameters are assumed as random variables. Prior PDFs for these variables, based on prior knowledge, are introduced in the analysis, see for instance Toni et al. (2009); Ritto (2014). Then, the PDFs are updated with the data obtained from the experiment. The technique is described next.

By hypothesis, the dynamics of the collision depends on the material properties. That is, the final generalized coordinates v and θ are functions of the constitutive parameters $\mathbf{\Theta}$. As mentioned before, the two variables were measured several times through a controlled experiment resulting in values v_f and θ_f; the data vector is $\mathbf{D} = (v_f, \theta_f)$. Then, an experimental PDF was generated for $\rho_{exp}(v_f, \theta_f) = \rho_{exp}(\mathbf{D})$. At the same time, given the constitutive parameters Θ,



(a)



(a) (b)

(b)

Figure 3: Experimental data. Left plots: Deflection speed (v_f); right plots: Deflection angle (θ_f). a) Histograms; b) PDFs obtained using the *ksdensity* function (Scrucca, 2001).

the integration of the system Eq. (1) gives us the general dynamics of the ball and, in particular, the final quantities $v_{f\,num}$ and $\theta_{f\,num}$ (or $\mathbf{D}_{num} = (v_{f\,num}, \theta_{f\,num})$). Therefore, the values of $\mathbf{D}_{num}$ depend on $\boldsymbol{\Theta}$, *i.e.*

$$\mathbf{D}_{num} = h(\boldsymbol{\Theta}) \tag{2}$$

Now, using the PDF $\rho_{exp}(\mathbf{D})$ given by the experiment with these numerical values, we propose the conditional probability function:

$$\rho(\mathbf{D}_{num} \mid \Theta) = \rho_{exp}(\mathbf{D}_{num}) = \rho_{exp}(h(\boldsymbol{\Theta})) \tag{3}$$

that is, the probability density to get $\mathbf{D}_{num}$ given $\boldsymbol{\Theta}$ (usually known as the likelihood function).

On the other hand, we assume a complete uncertainty of the parameters $\boldsymbol{\Theta}$, *i.e.* a uniform prior PDF of $\boldsymbol{\Theta}$: $\rho_{prior}(\boldsymbol{\Theta}) = uniform$, limited to the interval $\mu_0 \leq \mu \leq \mu_l$ and $\eta_0 \leq \eta \leq \eta_l$.

Using the Bayes formula, we can compute the posterior PDF of $\boldsymbol{\Theta}$

$$\rho_{post}(\boldsymbol{\Theta} \mid \mathbf{D}) = \frac{\rho(\mathbf{D}_{num} \mid \boldsymbol{\Theta})\rho_{prior}(\boldsymbol{\Theta})}{P_T(\mathbf{D})} \tag{4}$$

where $\rho_{post}(\boldsymbol{\Theta} \mid \mathbf{D})$ is the PDF of the vector of interest $\boldsymbol{\Theta}$ given data $\mathbf{D}$, *i.e.* the updated PDF, and $P_T(\mathbf{D})$ is the total probability of data $\mathbf{D}$, which gives the normalization constant:

$$P_T(\mathbf{D}) = \int \rho(\mathbf{D}_{num} \mid \boldsymbol{\Theta})\rho_{prior}(\boldsymbol{\Theta})\, d\boldsymbol{\Theta} \tag{5}$$

where $d\boldsymbol{\Theta} = d\mu d\eta$. This integral yields a normalization constant that makes the PDF norm equal to one. The ultimate goal is to get a PDF for every parameter of the vector $\boldsymbol{\Theta}$, *i.e.* a set of functions:

$$\rho_\eta(\eta \mid \mathbf{D}) \text{ and } \rho_\mu(\mu \mid \mathbf{D}) \tag{6}$$

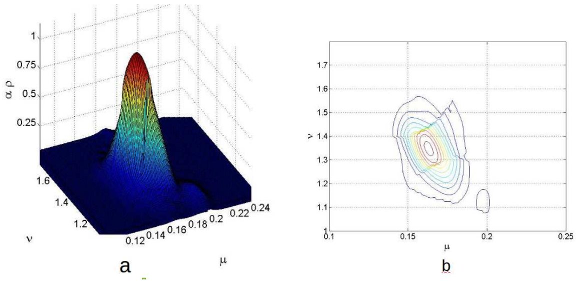


Figure 4: Posterior joint probability function $\rho_{post}(\mathbf{\Theta}|\mathbf{D})$. a) Surface function; b) contours.

and since the function $\rho_{post}(\mathbf{\Theta} \mid \mathbf{D})$ is the joint PDF of the two parameters, we use the marginal PDFs to find each probability function, as follows,

$$\begin{aligned} \rho_\eta(\eta \mid \mathbf{D}) &= \int \rho_{post}(\mathbf{\Theta} \mid \mathbf{D})\, d\mu \\ \rho_\mu(\mu \mid \mathbf{D}) &= \int \rho_{post}(\mathbf{\Theta} \mid \mathbf{D})\, d\eta \end{aligned} \tag{7}$$

The process to compute Eq.(5) can demand a large amount of CPU time. To speed up the numerical simulation, the Markov Chain Monte Carlo (MCMC)/ Metropolis–Hastings algorithm (Gamerman and Lopes, 2006) can be efficiently used to find the posterior distribution. These MCMC strategies are very attractive since the knowledge of the normalization constant becomes unnecessary and more, due to the simple implementation to get both the joint probability and the marginal integrals.

5 RESULTS

Once the experimental data described in Section 3 were obtained, the numeric model was run with the same geometrical, initial conditions and density values of the sphere under study. The equations of motion Eq. (1) was solved in Matlab using a Runge-Kutta method (Lasagni, 1988).

5.1 Parameter estimation

As explained in Section 4 and once the parameters $\mathbf{\Theta}$ are chosen, the whole dynamics of the impact can be reconstructed. However, it is important to remind that the values of $\mathbf{\Theta}$ are originally unknown. In order to estimate their values, it is necessary to calculate the marginal probability integrals given by Eq. (7). The computation of these integrals implies the evaluation of the dynamic model of the impact at as many points (in the $\mathbf{\Theta}$ space) as necessary to get acceptable values for the PDFs. In order to reduce the CPU time, the Metropolis-Hastings algorithm is implemented. In particular, the Matlab Metropolis-Hasting function was employed with 15000 samples, a symmetric proposal function, and the lag and burning parameters assumed to be 3 and 100, respectively. Figure 4 shows the posterior joint probability function $\rho_{post}(\mathbf{\Theta} \mid \mathbf{D})$ after a uniform prior PDF of $\mathbf{\Theta}$: $\rho(\mathbf{\Theta}) = uniform$ limited to the interval $0 \leq \mu \leq 0.5$ and $0 \leq \eta \leq 5$ kg/s, is assumed.

A comparison between the prior and the posterior marginal probability density functions for the variables considered in the analysis is presented in Figure 5. The prior is represented by dashes horizontal lines and the posterior is shown with full lines. It can be seen that the data used to update the prior added a great amount of information. Moreover, an uncertainty of 0.5 in the coefficient of friction is updated to 0.02 with 90% of confidence in the posterior PDF. In the case of viscosity coefficient, we get a 0.2 kg/s with the same confidence. The 90% confidence

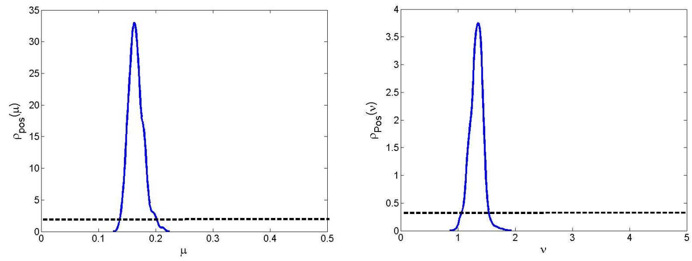


Figure 5: Posterior marginal probability density functions for the variables . The prior is in dashes lines and the posterior is shown in uniform lines.

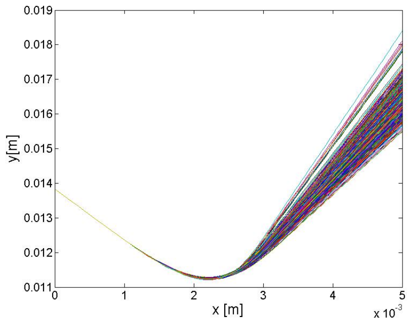


Figure 6: Uncertainty Quantification. Trajectory of the mass center of the ball in the collision problem. Curves of a subset of values of the Markov chain.

intervals of these parameters are:

$$\mu_{90\%} = 0.16 \pm 0.02; \qquad \eta_{90\%} = (1.35 \pm 0.2)kg/s. \tag{8}$$

Now that the posterior PDF is computed, the inverse problem is completed and the hypothesis of this article: *It is possible to indirectly measure the constitutive parameters of a rubber body knowing only the output angle and velocity of bouncing ball,* is true (in a statistical sense). The statistical values can be used to estimate other variables through computational simulations, *i.e.* a propagation of uncertainties. An illustration is included in the next subsection.

5.2 Stochastic simulation and direct problem

The physical experiment of Section 3 gave us the statistical (PDF) response of the kinematics of a bouncing ball. With this information, in the previous subsection we estimated the PDF of two unknown parameters, i.e. the friction coefficient μ and the viscosity η using the Bayes formula. In order to calculate the marginal PDF of these two parameters, a Markov Chain Monte Carlo (MCMC) algorithm was implemented. This Markov chain constitute a set of pairs of values compatible with the joint PDF illustrated in Figure 7. We can reconstruct the stochastic solution of the mechanical problem integrating the equations of motion (1) for each pair of the Markov chain. Figure 6 shows the resulting trajectory of the ball during the collision using a subset of the Markov chain. In this figure, the dispersion in the trajectory beyond the interaction with the rigid semi-space can be observed.

Since the complete solution of eq.(1) is available, it is possible evaluate the sensibility of the model to the stochastic parameters by a propagation of the uncertainties.The aim is to get the statistic of the bouncing velocity and angle to verify the procedure. Thus, the experiment is

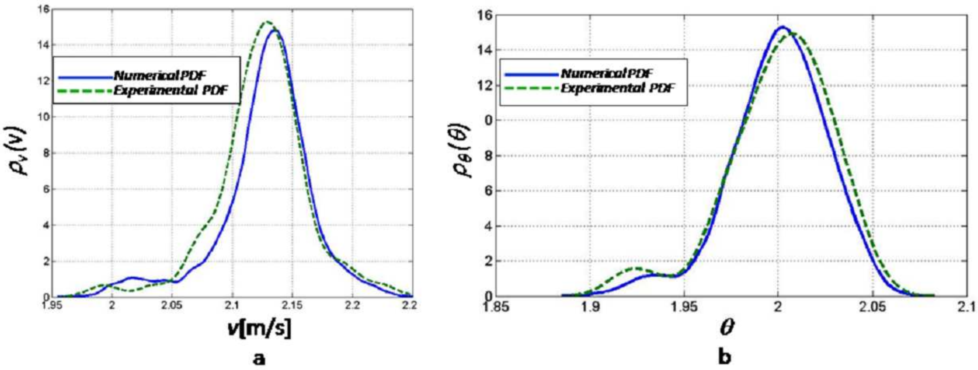


Figure 7: Uncertainty quantification. Reconstruction of the PDFs of the variables measured in the experiment. Full and dashed lines represent the numerical and experimental results, respectively. a) PDFs of the bouncing velocity; b) PDFs of the bouncing angle.

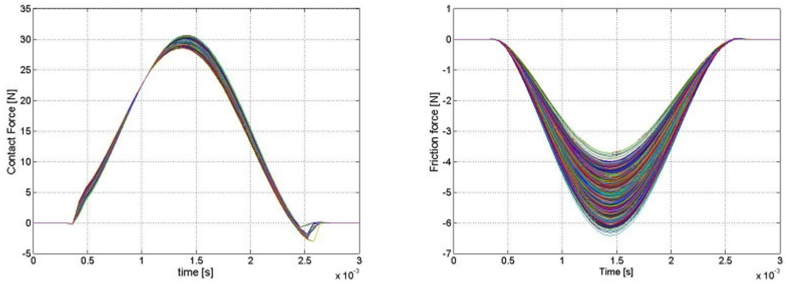


Figure 8: Uncertainty quantification. Statistical realizations of the forces involved in the interaction. a) Normal force; b) friction force.

numerically simulated in a direct problem using the estimated PDFs of the two parameters (friction and viscosity coefficients) as input data. The experimental results are thus reconstructed as a result of stochastic simulation. The PDFs of bouncing velocity and angle for both approaches (numerical and experimental) are shown in Figure 7.

Furthermore, some quantities of interest sometimes difficult to be measured directly, can be calculated with the propagation of the uncertainties derived of the Bayesian inference. From the numerical model and after introducing the PDFs of the parameters $\{\mu, \eta\}$, the friction and normal forces can be reproduced. Figure 8 a) depicts some realizations of the normal force during the collision. Both the shape and magnitudes are conserved and this variable does not seem to be sensitive to the random variability of the friction and viscous coefficients. On the other hand, the friction force (Figure 8 b) shape is preserved but its maximum magnitude exhibits strong variations.

Finally, the time of contact was analyzed from a statistical viewpoint. The resulting PDF is shown in Figure 9. It is observed that the range of the time of contact is very small and the variations are in the order of thousandths of a second. A bimodality is present in PDF.

6 CONCLUSIONS

The principal objective of this work is to solve the inverse problem using a Bayesian algorithm in order to estimate two constitutive parameters from the limited kinematic information collected in an experiment. This technique has two great advantages. One is the possibility of

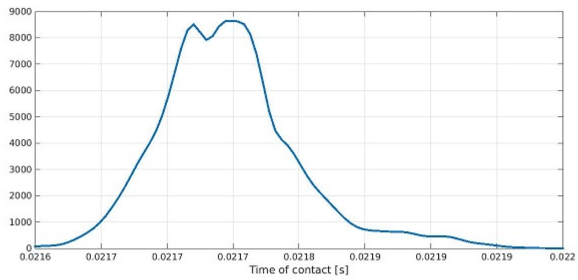


Figure 9: Uncertainty quantification. PDF of the time of contact.

performing a robust optimization and obtaining a probability distribution of the sought parameters and the other, the inference of constitutive parameters from indirect experiment data. The study addressed the collision of a viscoelastic ball with a rigid surface. Experimental tests were carried out and the trajectories of the mass center of the ball were recorded with a high-speed camera. Then, the data (rebound velocity and angle) were processed with a dedicated software. Using this information, the likelihood functions were then derived under certain hypotheses. After the application of the Bayesian inference, the best estimate of the constitutive parameters (friction and viscosity coefficients) were obtained after indirect measurements using a digital hi-speed recorder, given the priors of the sought parameters.

In order to verify the proposed methodology, the original experiment was numerically reconstructed with a stochastic simulation propagating the uncertainties. The agreement in the results was found to be more than acceptable. More, using the stochastic constitutive parameters, other quantities of interest were calculated. Thus, the normal and friction forces were found and the results indicate that the first one is not significantly affected by the variation of the random coefficients. On the other hand, the friction force magnitude is sensitive to the parameters variation. Also, the time of contact exhibits some particular features such as a bimodality in its PDF.

The methodology shows to provide a useful tool to solve an inverse problem with a stochastic approach which allows to reference the results in a statistic frame starting from indirect and sparse information.

At present, a more complex model within the mechanics of solids is being implemented to compare the results from the present elementary model. Also, the validation of the friction parameter value is being tackled through a physical friction experiment. The new results will be presented in the congress.

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