Asociación Argentina
de Mecánica Computacional

Mecánica Computacional Vol XXXVIII, págs. 309-318 (artículo completo)
H.G. Castro, J.L. Mroginski, R.R. Paz, M.A. Storti (Eds.)
Resistencia, 1-5 Noviembre 2021

STUDY OF NEW GUIDELINES FOR MATERIAL DISTRIBUTION IN VIBRATING AFG BEAMS

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Keywords: AFG beam, Bernoulli-Euler, Free vibrations, Rayleigh-Ritz, Dynamic stiffening.

Abstract. The free transverse vibrations of axially functionally graded (AFG) beams with two classic boundary conditions are studied. The distinctive qualities of this class of advanced materials acquire greater importance when the resistant element is in vibratory environments in which the inertial effect can be decisive. The classical Bernoulli-Euler theory is adopted to describe the flexural behavior of the beam. Due to the analytical difficulties, governing equations with variable coefficients, their solution is normally approached by approximate methods. This has motivated the subject to be approached mainly from a mathematical perspective, emphasizing the analysis on the numerical and analytical tools that lead to its solution. This mathematical approach also includes the distribution of the constituent materials. In the present work the composition of the FG material according to engineering criteria based on the structural behavior of the beam is proposed. The problem is solved by applying the approximate methods of Rayleigh-Ritz. A variety of numerical examples are evaluated with different variations in material composition. Dynamic stiffening effect is achieved and discussed. The results agree with particular situations of the model, available in the scientific literature.

1 INTRODUCTION

In recent years, the use of advanced materials whose properties vary gradually in some of its dimensions (FGM) has increased. Such materials were used for the first time by Japanese researchers who presented themselves as a thermal barrier material in the mid-1980s, Niino et al. (1987). In present paper, beams of materials whose properties vary functionally along the axis (AFG) are studied. A review of the literature reveals that most of the initial work on functionally graded beams has considered the gradation of material properties in the thickness direction. There have been far fewer researchers who considered the variation of material properties in the axial direction (AFG). Probably because the problem becomes more complicated as variable coefficients appear in the governing differential equations. This difficulty has led most of the studies have as main objective the development of mathematical and numerical tools for their solution. In the conformation of the FG material have been considered generally distributions that follow some mathematical concept: symmetric, asymmetric, parabolic, etc. In fact, the few analytical solutions obtained have been for AFG beams of arbitrary specific gradients: mention must be made of the thorough works of Professor Elishakoff and his colleagues, Elishakoff (2005), Wu, Wang and Elishakoff (2005)), making use of the semi-inverse method. Li et al. (2013) obtained closed form solutions for uniform AFG beams whose bending stiffness and distributed mass density are assumed to obey a unified exponential law. Among the numerous works that analysed the problem and proposed procedures for its solution, mention must be made of the following: Shahba and Rajasekaran (2012) studied longitudinal and transverse free vibration and buckling of AFG Bernoulli-Euler beams using the differential transform element method (DTEM) and differential quadrature element method of lowest-order (DQEL). Chen et al. (2017) introduced a numerical method to transform the differential equation into a set of linear algebraic equations with the displacement function being expanded using Taylor series or Chebyshev polynomials and Xie et al. (2017) by means of a spectral collocation approach based on integrated polynomials. Šalinić et al. (2018) presented a complete analysis for longitudinal and transverse vibrations of bars and beams AFG Bernoulli-Euler and solved them by means of the symbolic-numeric method of initial parameters (SNMIP). Cao et al. (2018) and Cao and Gao (2019) implemented the asymptotic development method (ADM). In the present paper, we study the influence of the distribution of constituent materials on the natural frequencies of Bernoulli-Euler beams with different boundary conditions. A material of high rigidity and low weight and another of lesser rigidity and greater weight are considered. The distribution criteria that are taken from the materials are not governed by purely mathematical laws, but are related to the structural behavior of the beams under study: e.g. bending moments, elastic deformation, modal forms. The well-known Rayleigh-Ritz method (Ilanko and Monterrubio, 2014) is employed and its suitability to apply to these types of problems has been demonstrated (Rossit et al. 2017, 2018), and the agreement of its results is verified with particular cases available in the literature.

2 ANALYTICAL APPROACH

In order to find the natural frequencies of the system, ω, one assumes that the beam deflection $v(\bar{x},t)$ may be expressed in the form:

$$v(\bar{x},t) = V(\bar{x})\mathrm{Cos}(\omega t) \quad (1)$$

According to the classical Bernoulli-Euler beam theory, the energy functional J for a vibrating beam of length L (see Figure 1) is given by:

$$2J\left[V(x)\right]=\int_0^1\left[E(x)I(x)\left(V''(x)\right)^2/L\right]dx-\omega^2\int_0^1\rho(x)A(x)L^3\left(V(x)\right)^2dx \tag{2}$$

where $x=\bar{x}/L$ is the dimensionless coordinate, $V(x)$ is the deflection, $A(x)$ is the cross section and $I(x)$ its second moment of area, $\rho(x)$ and $E(x)$ are the FGM density and Young´s modulus. $('')$ indicates the second derivative with respect to the spatial variable x.

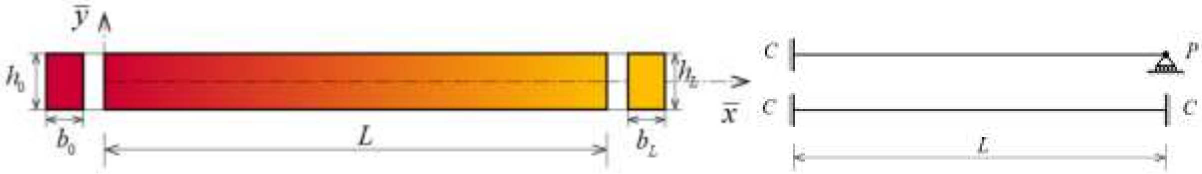


Figure 1: AFG beam with two different boundary conditions.

As the material and geometric characteristics of the beam may be general, one can define:

$$E(x)=E_0\,f_E(x);\,\rho(x)=\rho_0\,f_\rho(x);\;h(x)=h_0\,f_h(x);b(x)=b_0\,f_b(x) \tag{3}$$

with $A(x)=A_0\,f_A(x);I(x)=I_0\,f_I(x);\;f_A=f_b\times f_h$ and $f_I=f_b\times f_h^3$.

The subscript "0" refers to the cross section of the beam adopted as the reference section. Substituting Eq. (3) into Eq. (2), the functional can be expressed:

$$J\left[V(x)\right]=E_0I_0/(2L)\times\left\{\int_0^1 f_E(x)f_I(x)\left(V''(x)\right)^2dx-\Omega^2\int_0^1 f_\rho(x)f_A(x)\left(V(x)\right)^2dx\right\} \tag{4}$$

$$\Omega=\omega L^2\sqrt{\rho_0A_0/(E_0I_0)} \tag{5}$$

To apply the Ritz method, it is necessary to approximate the spatial component of the solution:

$$V(x)\cong V_a(x)=\sum_{j=1}^{N}C_j\varphi_j(x) \tag{6}$$

where φ_j are coordinate functions that satisfy the essential boundary conditions, C_j are arbitrary constants. N is the number of terms in the approximation.

Following Ritz' procedure, the functional is minimized with respect to every arbitrary constant:

$$\partial J\left[V_a(x)\right]/\partial C_j=0\;,\;j=1,2...,N \tag{7}$$

Then a linear system of equations is formed:

$$\mathbf{R}\left\{C_j\right\}=\left\{0\right\} \tag{8}$$

which results in the following eigenvalue equation:

$$\mathbf{R=K\text{ - }\Omega^2\,M} \tag{9}$$

$$k_{ij}=\int_0^1 f_Ef_I\,\varphi_i''\varphi_j''\,dx\;;\;m_{ij}=\int_0^1 f_\rho f_A\,\varphi_i\varphi_j\,dx \tag{10}$$

are the elements of matrices **K** and **M**, respectively.

Then, the eigenvalue problem can be expressed as:

$$\left|\mathbf{KM}^{-1}-\Omega^2\,\mathbf{I}\right|=\left|\mathbf{B}-\lambda\,\mathbf{I}\right|=0\;,\;\mathbf{B}=\mathbf{KM}^{-1}\;,\;\lambda=\Omega^2. \tag{11}$$

3 NUMERICAL RESULTS AND DISCUSSION

3.1 Comparative Study

First, the efficiency of the approach is checked by comparing its results with values available in the literature. Comparison is made with the exact results obtained by Li et al. (2013) for a particular case of stiffness and mass distribution:

$$E(x)=E_0\,e^{4x};\ \rho(x)=\rho_0\,e^{4x} \tag{12}$$

for the clamped-pinned (C-P) beam and the clamped-clamped (C-C) beam. In the present approach Eq. (6) for the C-P case is taken as:

$$\left\{\varphi_j(x)\right\}_{j=1}^{N}=\left\{(2x^2-5x+3)\,x^{j+1}\right\}_{j=1}^{N} \tag{13}$$

which verify the essential conditions and also the natural condition. And for the C-C case:

$$\left\{\varphi_j(x)\right\}_{j=1}^{N}=\left\{(x^2-2x+1)\,x^{j+1}\right\}_{j=1}^{N} \tag{14}$$

where all the boundary conditions are essential. Table 1 shows the accuracy of present results.

AFG	Ω_1	Ω_2	Ω_3	Ω_4	Solution
C-P beam	11.1828	48.2609	103.096	177.570	R-Ritz M.
	11.1828	48.2607	103.075	177.375	Li et al. (2013)
C-C beam	24.7896	64.7097	124.225	204.531	R-Ritz M.
	24.7896	64.7094	124.196	203.304	Li et al. (2013)

Table 1: Frequency coefficients for AFG beams (N=17).

3.2 Proposed cases

In the following, different cases of distribution of the constituent materials of an AFG beam will be analysed in order to evaluate the incidence of that distribution in its dynamic behavior, through the values of its natural frequencies.

In the calculations, the AFG material made of steel and aluminum oxide Al_2O_3 (alumina) proposed by Su et al. (2013) is used. Their Young modulus and density are:

$$E_{St}=210\,\text{GPa},\ \rho_{St}=7800\,\text{kg/m}^3;\ E_{Alum}=390\,\text{GPa},\ \rho_{Alum}=3960\,\text{kg/m}^3,\ \mu_{St}=\mu_{Alum}=0{,}3. \tag{15}$$

The relationships between material properties are: $E_{Alum}/E_{St}=1.857$ for Young's modulus and $\rho_{Alum}/\rho_{St}=0.508$ for the density. Note that the alumina, more rigid, is lighter than steel.

In the present study, it is proposed to study two cases of boundary conditions (C-P beam and C-C beam) analyzing material distributions that follow laws related to the structural behavior of the beams.

As it was said, in general, the material distribution of the vibrating AFG beam has been approached from a mathematical perspective rather than an engineering concept. That is why in the descriptions of the variation of the constituent materials, terms such as symmetric, asymmetric, parabolic predominate. As is known, the dynamic behavior of a resistant structure depends on the relationship between the elastic deformation energy and the kinetic energy. Increasing that ratio, natural frequencies of the structure rise. This is important, especially in the case of the first natural frequency, to move it away from the usual frequencies that affect the structures and avoid the dangerous situation of resonance. Therefore, distributions will be considered where a larger proportion of alumina will be placed in the sectors where the flexural effort is greater (to increase the elastic energy) and in those in which the displacement is greater (to decrease the kinetic energy), Figure 2.

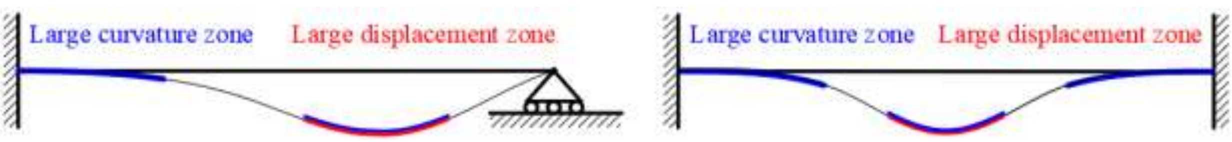


Figure 2: First modal shape of a C-P and a C-C beam.

In all evaluated cases, in order to facilitate comparisons, the weight of the beam is considered in relation to the uniform steel beam, which is taken as reference material W_b; and consequently, the frequency coefficient Ω is taken as:

$$\Omega = \omega L^2 \sqrt{\rho_{st} A_0 / \left(E_{st} I_0 \right)} \text{ and } W_b = g \int_0^L \rho\, b\, h\, dx / \rho_{st}\, g\, b_0 h_0 L \tag{16}$$

3.2.1 Clamped-pinned beams (C-P)

The first two cases studied are the classic variations of the constituent materials in the AFG beams considered in the scientific literature (Shahba et al. (2011), Gilardi et al. (2018)).

Case 1: Asymmetric polynomial variation law.

A generic material proper ty $R(x)$ is assumed to vary along the beam axis *x*:

$$\text{1a) } R(x) = R_{St} + \left(R_{Alum} - R_{St} \right) x^n \text{ ; 1b) } R(x) = R_{Alum} + \left(R_{St} - R_{Alum} \right) x^n \tag{17}$$

where *n* is the inhomogeneity parameter, with $n \geq 0$.

Case	n	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	W_b
1a)	0.5	23.5503	76.1715	158.911	271.753	414.696	0.672
	1	20.9379	68.0893	142.275	243.45	371.612	0.754
	2	18.5267	60.9005	127.561	218.473	333.634	0.836
1b)	0.5	18.9174	61.3303	128.098	219.208	334.656	0.836
	1	20.9028	68.0117	142.168	243.326	371.476	0.754
	2	23.5626	76.2409	158.888	271.53	414.186	0.672

Table 2: Frequency coefficients for a C-P beam of AFG material – Case 1.

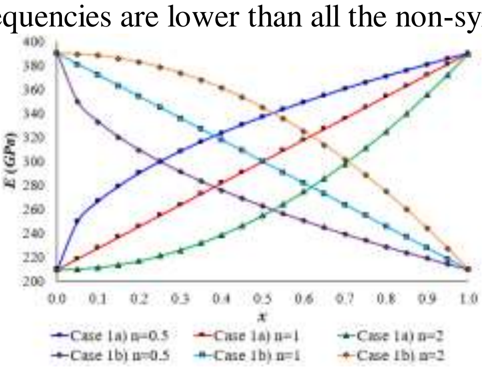
For the AFG Clamped-Pinned beam, Case 1, Figure 3.A) presents the variation of Young´s modulus and Figure 3.B) shows the variation of mass density along the beam axis *x*.

Table 3 presents the first five frequency coefficients obtained for the distribution laws of Eq. (18). The calculations have been made for $N = 20$, as for all remaining cases.

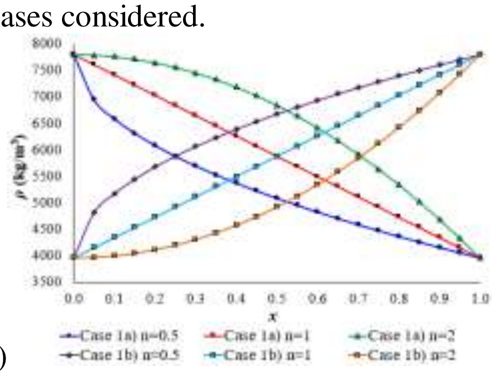
Case 2: Symmetric quadratic variation law. Again, two compositions are considered:

$$\text{2a) } R(x) = R_{St} + 4\left(R_{Alum} - R_{St} \right)\left(x - x^2 \right) \text{; 2b) } R(x) = R_{Alum} + 4\left(R_{St} - R_{Alum} \right)\left(x - x^2 \right) \tag{18}$$

Figures 4.A) and 4.B) show the variation of Young´s modulus and mass density for the AFG Clamped-Pinned beam, Case 2, along the beam axis *x*. Table 3 shows the first five frequency coefficients for the distribution laws of Eq. (18). As can be seen, the symmetric distribution raises all frequency values when the distribution of the material is such that the central area of the beam contains higher alumina proportions. The increase is greatest for the first frequency. On the other hand, when the central zone consists mostly of steel, all frequencies are lower than all the non-symmetric cases considered.



A)



B)

Figure 3: AFG Clamped-Pinned beam. Case 1: A) Young´s modulus B) mass density.

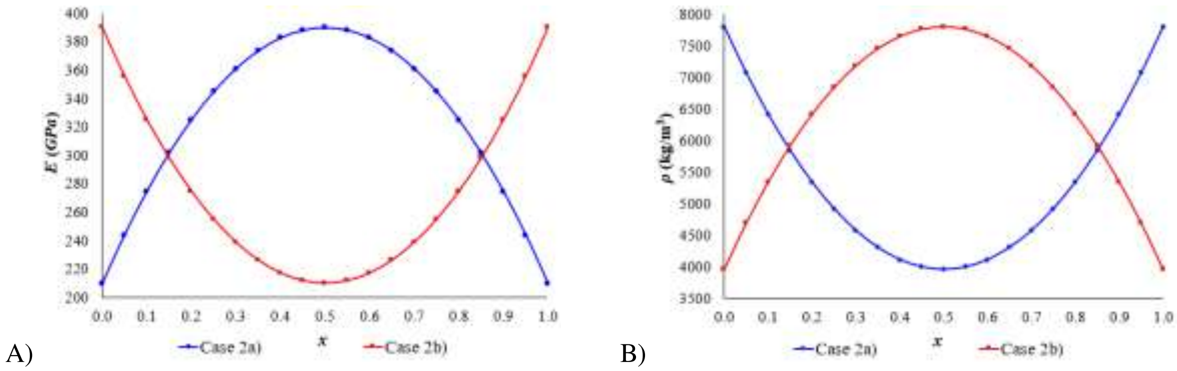


Figure 4: AFG Clamped-Pinned beam. Case 2: A) Young´s modulus; B) mass density.

Case	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	W_b
2a)	25.2453	77.5621	160.062	272.557	415.124	0.672
2b)	17.9158	59.934	126.249	216.904	331.871	0.836

Table 3: Frequency coefficients for a C-P beam of AFG material – Case 2.

It is also of interest to note that in case 2a), the weight of the beam coincides with the smallest of all the variants taken into account.

From now on, distributions of the constituent materials governed by laws related to the structural behavior of the beam will be considered:

Case 3: Distribution of the constituent materials with the proportion of alumina following the law of variation of the bending moment caused by a uniform distributed load

$$R(x) = R_{St} + (R_{Alum} - R_{St})\left|1 - 5x + 4x^2\right| \tag{19}$$

where, it is appropriate to remember, $|\ |$ represents absolute value.

With this distribution law, it is not possible that in the section of maximum positive bending moment the cross section is completely constituted by alumina.

To achieve this, it is necessary to normalize the expression giving rise to the following case analysed as Case 4.

Case 4: Distribution of the constituent materials with the proportion of alumina following the normalized law of variation of the bending moment caused by a uniform distributed load

$$R(x) = R_{St} + (R_{Alum} - R_{St})\left|(1 - 5x + 4x^2)(1 + (56/45)\,x)\right| \tag{20}$$

Case 5: The proportion of alumina in the composition is proportional to the transverse displacement of the first modal form, normalizing the maximum:

$$R(x) = R_{St} + (R_{Alum} - R_{St})\, y(x)/y_{máx} \tag{21}$$

$$\begin{gathered} y(x) = \mathrm{Cosh}(kx) - \mathrm{Cos}(kx) - \alpha\,\mathrm{Sinh}(kx) + \alpha\,\mathrm{Sin}(kx)\ ,\ y_{máx} = y(0.58079) = 1.50940, \\ k = 3.92738 \text{ and } \alpha = \left[\mathrm{Cosh}(k) - \mathrm{Cos}(k)\right]/\left[\mathrm{Sinh}(k) - \mathrm{Sin}(k)\right] \end{gathered} \tag{22}$$

Case 6: The proportion of alumina in the composition follows the law of variation of the bending moment caused by a load equivalent to the first modal shape

$$R(x) = R_{St} + (R_{Alum} - R_{St})\left|g_1(x)/g_1(0)\right| \tag{23}$$

$$g_1(x) = \int\left(\int y(x)\,dx\right)dx - 0.00021x + 0.00007\ ,\ \ g_1(0) = 0.12973\,. \tag{24}$$

Again, it is necessary to normalize the expression so that the section of the maximum positive bending moment is entirely alumina.

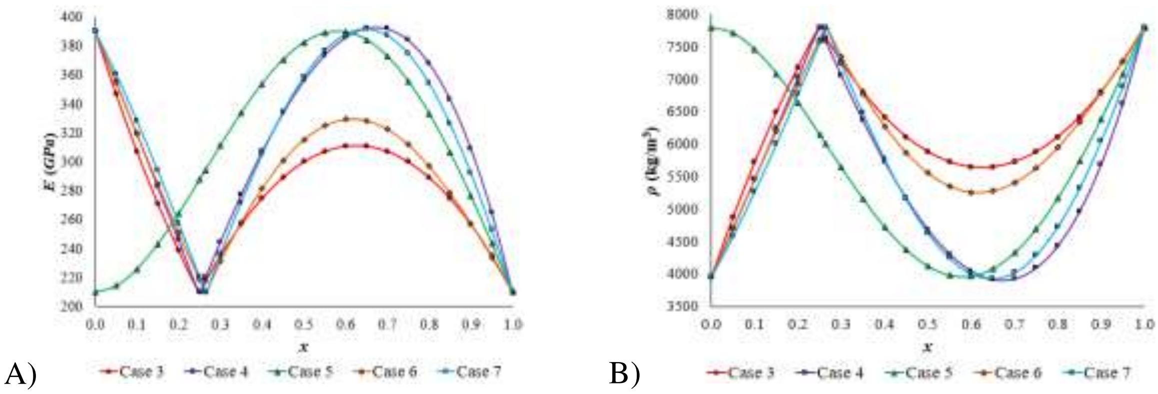


Figure 5: AFG Clamped-Pinned beam. Cases 3 to 7: A) Young´s modulus; B) mass density.

<u>Case 7</u>: The proportion of alumina in the composition follows the normalized law of variation of the bending moment caused by a load equivalent to the first modal shape

$$R(x) = R_{St} + \left(R_{Alum} - R_{St}\right)\left|g_1(x)(6.33586x + 7.70804)\right| \tag{25}$$

The distributions of Young's modulus and the mass density of the composition of Clamped-Pinned are indicated in Figure 5 A) and B), for cases 3 to 7. The first five frequency coefficients, for each of the cases described above, are shown in Table 4. Observing the values in Table 5, it is noted that the objective of increasing the first frequency is achieved in cases 4 and 7, although in a non-significant way with respect to case 2a). Even the weight increases slightly. In order to evaluate the dynamic stiffening that is achieved with the use of FG materials, Table 5 shows the first five frequency coefficients for a Clamped-Pinned homogeneous steel beam ($W_b = 1$) (Karnovsky and Lebed, 2000). It is clear that all the cases considered raise the frequency values and decrease the weight of the beam, achieving efficient stiffening.

Case	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	W_b
3	21.2089	63.7631	132.410	228.541	349.494	0.805
4	25.5231	73.8760	153.438	265.237	404.041	0.692
5	24.3564	72.2013	149.694	255.174	358.025	0.719
6	22.0703	65.1126	136.337	235.826	359.713	0.782
7	25.4014	72.5072	152.461	263.419	400.232	0.698

Table 4: Frequency coefficients for an AFG beam with Clamped-Pinned ends – Cases 3 to 7.

Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
15.4182	49.9649	104.248	178.270	272.031

Table 5: Frequency coefficients for a homogenous beam with Clamped-Pinned ends.

Stiffening efficiency η_i is defined as the following coefficient (Laura et al., 2001):

$$\eta_i = \left[\Omega_i\,(\mathrm{AFG}) / \Omega_i\,(St)\right] / W_b \tag{26}$$

Table 6 shows the stiffening coefficients η_1 obtained for the first frequency in all the cases analysed. The highest levels of stiffening efficiency correspond to cases 2a), 4 and 7, achieving case 4 and case 7 the highest values of the first frequency.

3.2.2 Clamped-Clamped AFG beam (C-C)

The distributions of Young's modulus and the density of the composition of each case are indicated in Figure 6 A) and B).

Case 1, Eq. (17)	n		0.50	1	2
	1a) *St-Alum*		2.273	1.801	1.437
	1b) *Alum-St*		1.468	1.798	2.274
Case 2, Eq. (18)	2a) *St-Alum-St*		2.437		
	2b) *Alum-St-Alum*		1.390		
Cases 3 to 7, Eqs. (19)-(25)	Case 3	Case 4	Case 5	Case 6	Case 7
	1.709	2.393	2.196	1.831	2.361

Table 6: Frequency coefficients for an AFG beam with Clamped-Pinned ends – Cases 1 to 7.

Case 8: First, the two classic quadratic symmetric distributions of the constituent materials described in the Eq. (18) are considered. Non-symmetric cases - Eq. (17) - are not taken into account by virtue of the symmetry of the C-C beam.

Again, distributions of the constituent materials governed by laws related to the structural behavior of the beam will be taken into account. Based on what was observed for the C-P beam, just normalized expressions are used:

Case 9: Distribution of the constituent materials with the proportion of alumina following the normalized law of variation of the bending moment caused by a uniform distributed load

$$R(x)=R_{St}+\left(R_{Alum}-R_{St}\right)\left|\left(1-6x+6x^2\right)\left(1+4x-4x^2\right)\right| \tag{27}$$

Case 10: The proportion of alumina in the composition is proportional to the transverse displacement of the first modal form, normalizing the maximum

$$R(x)=R_{St}+\left(R_{Alum}-R_{St}\right)y(x)/y_{máx} \tag{28}$$

$$\begin{gathered} y(x)=\mathrm{Cosh}(kx)-\mathrm{Cos}(kx)-\alpha\,\mathrm{Sinh}(kx)+\alpha\,\mathrm{Sin}(kx)\;,\; y_{máx}=y(0.50)=1.58815, \\ k=4.73004 \;\text{ and }\; \alpha=\left[\mathrm{Cos}(k)-\mathrm{Cosh}(k)\right]/\left[\mathrm{Sin}(k)-\mathrm{Sinh}(k)\right] \end{gathered} \tag{29}$$

Case 11: The proportion of alumina in the composition follows the normalized law of variation of the bending moment caused by a load equivalent to the first modal shape

$$R(x)=R_{St}+\left(R_{Alum}-R_{St}\right)\left|g_2(x)\;c(x)\right| \tag{30}$$

$$g_2(x)=\int\left(\int y(x)dx\right)dx+(3.84230x-1.28077)\times10^{-30},\; c(x)=-28.8713x^2+28.8713x+11.1866 \tag{31}$$

The first five frequency coefficients, for each of the cases described above, are shown in Table 7.

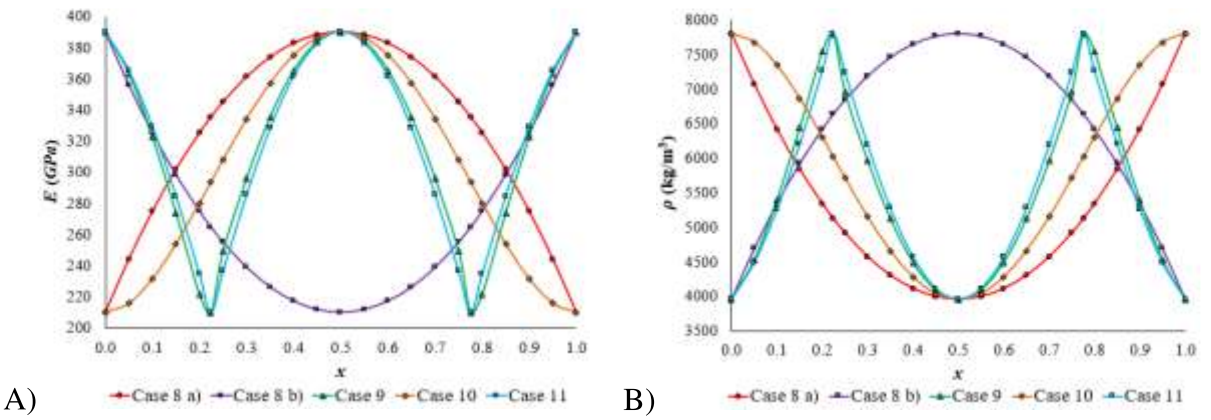


Figure 9: AFG Clamped-Clamped beams. Cases 8 to 11: A) Young´s modulus; B) mass density.

Again, it is observed that the cases corresponding to the distribution of the material following the law of bending moments - cases 9 and 11 -, show the highest coefficients for the first vibration frequency. In order to evaluate the dynamic stiffening that is achieved, Table 8 shows the first five frequency coefficients for a Clamped-Clamped homogeneous steel beam ($W_b = 1$) (Karnovsky and Lebed, 2000).

Case	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	W_b
8 a)	35.6800	95.1011	184.971	304.885	454.898	0.672
8 b)	26.7892	74.3683	146.558	243.111	364.011	0.836
9	36.5655	86.1039	170.108	265.123	289.068	0.699
10	33.3763	85.7577	167.725	276.561	413.111	0.742
11	36.2679	84.8553	170.345	290.331	438.783	0.701

Table 7: Frequency coefficients for an AFG beam with Clamped-Clamped ends – Cases 8 to 11.

Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
22.3733	61.7061	120.903	199.859	298.556

Table 8: Frequency coefficients for a homogenous beam with Clamped-Clamped ends.

Case	8 a)	8 b)	9	10	11
η_1	2.374	1.432	2.337	2.009	2.312

Table 9: Coefficients η_1 for an AFG beam with Clamped-Clamped ends – Cases 8 to 11.

In the same way as for the AFG C-P beams, the stiffening efficiency coefficient η_i, Eq. (26), is calculated for the AFG C-C beams. Table 9 shows this coefficient for the first natural frequency. Similar to what happened with the C-P beam, the cases in which the constituent materials have a quadratic symmetric distribution (case 8) or are distributed following bending stress laws (cases 9 and 11) are those with the best level of stiffening efficiency. Again, the highest levels of the first transverse vibration frequency are obtained in cases 9 and 11.

4 CONCLUSIONS

The structural advantages of the use of FG materials in the design of beams have been ratified. In fact, they allow the development of singular value performances in vibratory environments, which have an interest in numerous technological applications of civil engineering, electronics, spatial engineering, etc., when stiffness and low weight are sought.

In the investigations on the subject, the distributions of the constituent materials have been considered following laws elaborated with mathematical criteria. In this document, material distributions are proposed that follow laws which represent the variation of some parameter related to the structural behavior of the beam. With this, the first transverse vibration frequency has risen, removing the structural element from the damage caused by those actions rich in low frequency components. The authors are confident that the use of preponderantly structural criteria in the distribution of the components of the FG materials will contribute to further increase the remarkable performance of the resistant structures constructed with these materials, enhancing their revolutionary impact in that broad field of engineering.

Once again, the classic Rayleigh-Ritz method demonstrates its versatility and precision to solve elasto-mechanics problems.

ACKNOWLEDGMENTS

The study has been sponsored by the Universidad Nacional del Sur, CONICET and the

Comisión de Investigaciones Científicas, Buenos Aires Province (CIC).

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