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A NUMERICAL APPROACH TO EFFECTIVE VISCOELASTIC PROPERTIES OF FIBER COMPOSITES

G. Koval Jr. [*], S. Maghous [†], G. J. Creus [*]

[*] CEMACOM-UFRGS
e-mail: creus@ufrgs.br

[†] ENPC, Paris
e-mail: maghous@lmsgc.enpc.fr

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Abstract. *Composite materials of polymeric base show considerable time dependence in their mechanical properties. In the present work, the effective viscoelastic behavior of one-directional fiber composites, based on components properties and geometry, is determined.*
The numerical approach proposed is based upon the theory of periodic homogenization - Sanchez-Palencia. We use expansions of the periodic component of displacements by means of Fourier series or alternatively Chebyshev polynomials (modified in order to fulfill periodic conditions).
The viscoelastic localization problem is solved as a sequence of elastic problems with initial viscoelastic strains. The solution of each elastic problem is achieved through minimization of a functional with respect to the coefficients of the displacement expansions (Fourier or Chebyshev).
Results obtained for the elastic case agree with solutions in the literature. Viscoelastic results are compared with analytical and numerical (finite element) results.
The advantage of the proposed method is its generality. In particular, it may be used with no fundamental changes for aging viscoelastic materials.

1 INTRODUCTION

In composite materials engineering, the material is designed concurrently with the structure. There are as many varieties as the designer needs or can devise. Thus, the determination of constitutive relation by physicals tests, usual for traditional materials is impractical. Micromechanics allows the designer to represent a heterogeneous material as an equivalent homogeneous material, usually anisotropic. There are several approaches, increasingly complex to derive micromechanical results. In this work, the determination of constitutive relations for elastic and viscoelastic fiber composites is tackled using Homogenization Theory - Sanchez-Palencia[9].

To this effect, the localization problem for the basic cell is solved representing displacements by means of analytical expansions (Fourier series and modified - Chebyshev polynomials) and determining the coefficients of the homogenized material by a minimization of the elastic energy over the domain. All the process is carried on using the Matlab software.

The elastic results are validated by comparison with known results – Barbero[1]; it is seen that Chebyshev polynomials provide good results with more efficiency.

The procedure is then extended to linear viscoelastic composite materials, where the minimization process is applied incrementally to a series of elastic localization problems with initial strain. The initial strain is the incremental time-dependent strain determined by means of a state variables procedure.

The content of the paper is as follows. In Section 2 the localization problem in elasticity is set. In Section 3 it is solved for the case of an elastic fiber composite and in Section 4 the procedure is extended to the viscoelastic case. In Sections 5 and 6, examples and comparison with other results are included.

2 SOLUTION OF THE LOCALIZATION PROBLEM IN ELASTICITY

2.1 Elementary cell

Heterogeneous materials with small scale periodic structure are considered. The corresponding elementary cell represents the smaller volume that completely describes the structure, Fig.1.

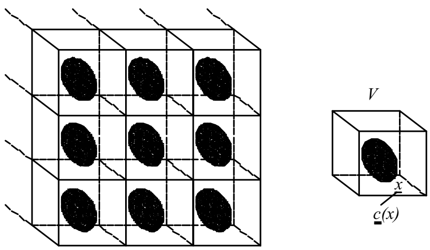

Figure 1: Material with periodical heterogeneities and the corresponding elementary cell

2.1.1 Field structure

Let $\underline{\sigma}$ be an statically admissible stress field (S.A.) and $\underline{u}$ a cinematically admissible displacements field. (C.A.):

$$\underline{\sigma} \text{ such that } div(\underline{\sigma}) = 0 \text{ and } \underline{\sigma} \cdot \underline{n} \text{ anti-periodic} \tag{1}$$

$$\underline{u} \text{ such that } \underline{u} = \underline{E}.x + \underline{u}^{*} \text{ , } \underline{u}^{*} \text{ periodic} \tag{2}$$

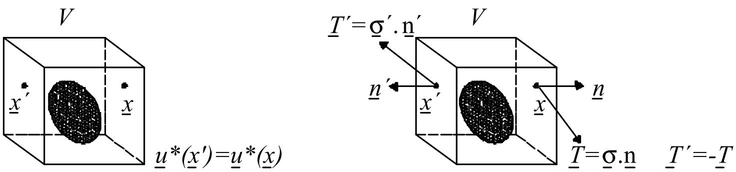

Figure 2: Periodicity of the displacement field and anti-periodicity of the stress field.

The strain tensor is

$$\underline{\varepsilon} = \underline{E} + \underline{\varepsilon}^{*} \text{ with } \underline{\varepsilon}^{*} = \frac{1}{2}\left(grad\underline{u}^{*} + (grad\underline{u}^{*})^{T}\right) \text{ - periodic part} \tag{3}$$

Because of periodicity

$$\langle \underline{\varepsilon} \rangle = \underline{E} + \langle \underline{\varepsilon}^{*} \rangle = \underline{E} \tag{4}$$

where $\langle . \rangle = \frac{1}{V}\int_{V} (.) dV$ represents the volumetric average over the cell domain V.

Eq. (4) indicates that $\underline{E}$ is the macroscopic strain tensor applied to the unit cell. The macroscopic stress tensor $\underline{\Sigma}$ is defined as

$$\underline{\Sigma} = \langle \underline{\sigma} \rangle \tag{5}$$

Hill[7]′s Lemma Let us $\underline{u}$ and $\underline{\sigma}$ be admissible displacement and stress fields. Then the average of the microscopic and microscopic virtual works are identical:

$$\langle \underline{\sigma} : \underline{\varepsilon} \rangle = \langle \underline{\sigma} \rangle : \langle \underline{\varepsilon} \rangle \tag{6}$$

2.2 Localization problem

The local constitutive law is

$$\underline{\sigma}(\underline{x}) = \underline{\underline{c}}(\underline{x}) : \underline{\varepsilon}(x) \quad \forall \underline{x} \in V \tag{7}$$

where the fourth order tensor $\underline{\underline{c}}$ is the elastic stiffness.

The determination of the macroscopic elastic law for the periodically heterogeneous material reduces to the solution of a boundary problem called localization problem, defined for the unit cell.

2.2.1 Displacement approach

Let $\underline{E}$ be a given constant symmetric tensor. The localization problem consists in the determination of the stress and displacement fields ($\underline{\sigma}$ e $\underline{u}$):

$$\begin{gathered}\underline{\sigma} \text{ such that } div(\underline{\sigma}) = 0 \text{ and } \underline{\sigma}\cdot n \text{ anti-periodic}\\ \underline{u} \text{ such that } \underline{u} = \underline{E}.x + \underline{u}^{*},\ \underline{u}^{*} \text{ and periodic}\\ \underline{\sigma} = \underline{\underline{c}} : \underline{\varepsilon}\end{gathered} \tag{8}$$

Once solved this problem, the constitutive macroscopic law that relates $\underline{\Sigma}$ with $\underline{E}$ (as defined in (5) and (4)) follows.

Because of linearity

$$\underline{\varepsilon} = \underline{\underline{A}} : \underline{E} \tag{9}$$

where: $\underline{\underline{A}} = \underline{\underline{A}}(\underline{x})$ represents the strain localization tensor. $\underline{\underline{A}}(\underline{x})$ depends only on geometrical and mechanical parameters and is independent from $\underline{E}$. Substituting (9) into (4):

$$\langle\underline{\varepsilon}\rangle = \langle\underline{\underline{A}} : \underline{E}\rangle = \langle\underline{\underline{A}}\rangle\underline{E} = \underline{E} \tag{10}$$

and recalling that de $\langle\underline{\underline{A}}\rangle = \underline{\underline{I}}$, where $\underline{\underline{I}}$ is the fourth order identity tensor.

$\underline{\sigma}$ and $\underline{\varepsilon}$ are related through the constitutive local equation

$$\underline{\sigma} = \underline{\underline{c}} : \underline{\varepsilon} = \underline{\underline{c}} : \underline{\underline{A}} : \underline{E} \tag{11}$$

The average of (11)

$$\langle\underline{\sigma}\rangle = \underline{\Sigma} = \langle\underline{\underline{c}} : \underline{\underline{A}} : \underline{E}\rangle = \langle\underline{\underline{c}} : \underline{\underline{A}}\rangle : \underline{E} = \underline{\underline{C}}^{\text{hom}} : \underline{E} \tag{12}$$

leads to the macroscopic elasticity relation

$$\underline{\underline{C}}^{\text{hom}} = \langle\underline{\underline{c}} : \underline{\underline{A}}\rangle \tag{13}$$

The symmetry of $\underline{\underline{C}}^{\text{hom}}$, that is not apparent from (13), can be found using Hill´s Lemma. Substituting (12) into $\underline{\Sigma} : \underline{E}$:

$$\underline{\Sigma} : \underline{E} = \underline{E} : \underline{\underline{C}}^{\text{hom}} : \underline{E} \tag{14}$$

From (6), (14) equals $\langle\underline{\sigma} : \underline{\varepsilon}\rangle$, and then

$$\langle \underline{\sigma} : \underline{\varepsilon} \rangle = \langle \underline{\varepsilon} : \underline{\underline{c}} : \underline{\varepsilon} \rangle = \langle \underline{E} : \underline{\underline{A}}^T : \underline{\underline{c}} : \underline{\underline{A}} : \underline{E} \rangle = \underline{E} : \langle \underline{\underline{A}}^T : \underline{\underline{c}} : \underline{\underline{A}} \rangle : \underline{E} \tag{15}$$

From (14) e (15):

$$\underline{\underline{C}}^{\hom} = \langle \underline{\underline{A}}^T : \underline{\underline{c}} : \underline{\underline{A}} \rangle \tag{16}$$

2.2.2 Stress approach

With a similar procedure for stress, we can obtain the expression

$$\underline{\underline{D}}^{\hom} = \langle \underline{\underline{B}}^T : \underline{\underline{d}} : \underline{\underline{B}} \rangle \tag{17}$$

The equivalence between the both approaches is guaranteed by the equality

$$\underline{\underline{C}}^{\hom} : \underline{\underline{D}}^{\hom} = I \tag{18}$$

2.3 Macroscopic elastic potential and minimization principle

Locally, for $\forall \underline{x} \in V$ the hyperelastic relation is

$$\underline{\sigma} = \frac{\partial \Psi(\underline{\varepsilon})}{\partial \underline{\varepsilon}} \tag{19}$$

where Ψ is the elastic potential. (convex in $\underline{\varepsilon}$).

Now, let $\underline{u}$, $\underline{\varepsilon}$ and $\underline{\sigma}$ be solutions for the localization problem for the macroscopic strain $\underline{E}$. And $\underline{u}'$ another displacement field admissible with $\underline{E}$:

$$\underline{u}' = \underline{E}.x + \underline{u}'^{*} ,\ \underline{u}'^{*} \text{ is periodic} \tag{20}$$

Eq. (20) can be written alternatively

$$\underline{u}' = \underline{u} + \underline{\delta u}' \ ;\ \underline{\delta u}' = \underline{u}'^{*} - \underline{u}^{*} \text{(periodic)} \tag{21}$$

Using the convexity condition for Ψ, we have

$$\Psi(\underline{\varepsilon}(\underline{u}')) - \Psi(\underline{\varepsilon}(\underline{u})) \geq \frac{\partial \Psi(\underline{\varepsilon}(\underline{u}))}{\partial \underline{\varepsilon}} : \left(\underline{\varepsilon}(\underline{u}') - \underline{\varepsilon}(\underline{u})\right) \tag{22}$$

or

$$\Psi(\underline{\varepsilon}(\underline{u}')) - \Psi(\underline{\varepsilon}(\underline{u})) \geq \frac{\partial \Psi(\underline{\varepsilon}(\underline{u}))}{\partial \underline{\varepsilon}} : \underline{\varepsilon}(\delta \underline{u}') \tag{23}$$

As $\dfrac{\partial \Psi(\underline{\varepsilon}(\underline{u}))}{\partial \underline{\varepsilon}} = \underline{\sigma}$ is the solution for the localization problem, it is admissible . Taking the average of (23) and applying Hill´s Lemma to the right side:

$$\left\langle \Psi(\underline{\varepsilon}(\underline{u}')) - \Psi(\underline{\varepsilon}(\underline{u})) \right\rangle \geq \left\langle \frac{\partial \Psi(\underline{\varepsilon}(\underline{u}))}{\partial \underline{\varepsilon}} : \underline{\varepsilon}(\delta \underline{u}') \right\rangle \tag{24}$$

$$\left\langle \frac{\partial \Psi(\underline{\varepsilon}(\underline{u}))}{\partial \underline{\varepsilon}} : \underline{\varepsilon}(\delta \underline{u}') \right\rangle = \left\langle \underline{\sigma} : \underline{\varepsilon}(\delta \underline{u}') \right\rangle = \left\langle \underline{\sigma} \right\rangle : \left\langle \underline{\varepsilon}(\delta \underline{u}') \right\rangle \tag{25}$$

As $\left\langle \underline{\varepsilon}(\delta \underline{u}') \right\rangle = 0$ due to periodicity of $\delta \underline{u}'$, we have for (24):

$$\left\langle \Psi(\underline{\varepsilon}(\underline{u}')) \right\rangle \geq \left\langle \Psi(\underline{\varepsilon}(\underline{u})) \right\rangle \tag{26}$$

for $\forall \underline{u}'$ cinematically with $\underline{E}$. From (26) we have, then

$$\left\langle \Psi(\underline{\varepsilon}(\underline{u})) \right\rangle = \min \left\{ \left\langle \Psi(\underline{\varepsilon}(\underline{u}')) \right\rangle / u' \;\; C.A.\; com \;\; \underline{E} \right\} \tag{27}$$

Thus, the desired solution minimizes the localization functional $\left\langle \Psi(\underline{\varepsilon}(\underline{u})) \right\rangle$.

2.3.1 Linear elastic case

In the linear elastic case:

$$\Psi(\underline{\varepsilon}) = \frac{1}{2} \underline{\varepsilon} : \underline{\underline{c}} : \underline{\varepsilon} = \frac{1}{2} \underline{\sigma} : \underline{\varepsilon} \tag{28}$$

Introducing the concept of macroscopic elastic potential $\Psi^{\hom}$:

$$\Psi^{\hom}(\underline{E}) = \left\langle \Psi(\underline{\varepsilon}(\underline{u})) \right\rangle = \min \left\{ \left\langle \Psi(\underline{\varepsilon}(\underline{u}')) \right\rangle / u' \;\; C.A.\; com \;\; \underline{E} \right\} \tag{29}$$

and substituting (28) into (29):

$$\Psi^{\hom}(\underline{E}) = \left\langle \frac{1}{2} \underline{\varepsilon} : \underline{\underline{c}} : \underline{\varepsilon} \right\rangle = \frac{1}{2} \left\langle \underline{\sigma} : \underline{\varepsilon} \right\rangle \tag{30}$$

Applying Hill´s Lemma in (30):

$$\Psi^{\hom}(\underline{E}) = \frac{1}{2} \left\langle \underline{\sigma} : \underline{\varepsilon} \right\rangle = \frac{1}{2} \left\langle \underline{\sigma} \right\rangle : \left\langle \underline{\varepsilon} \right\rangle = \frac{1}{2} \underline{\Sigma} : \underline{E} \tag{31}$$

Defining the tensor that relates $\underline{\Sigma}$ and $\underline{E}$ ($\underline{\underline{C}}^{\hom}$) as

$$\underline{\Sigma} = \underline{\underline{C}}^{\hom} : \underline{E} \tag{32}$$

Substituting (32) in (31):

$$\Psi^{\hom}(\underline{E}) = \frac{1}{2} \underline{E} : \underline{\underline{C}}^{\hom} : \underline{E} \tag{33}$$

We observe that Ψ^{hom} satisfies the relation equivalent to (19) for macroscopic fields:

$$\frac{\partial \Psi^{\text{hom}}}{\partial \underline{E}} = \underline{\underline{C}}^{\text{hom}} : \underline{E} = \underline{\Sigma} \tag{34}$$

Then, we can say that the macroscopic elastic potential is the minimum of the average of the microscopic potential over the unit cell:

$$\Psi^{\text{hom}}(\underline{E}) = \frac{1}{2}\underline{E} : \underline{\underline{C}}^{\text{hom}} : \underline{E} = \min \left\{ \left\langle \Psi(\underline{\varepsilon}(\underline{u}')) \right\rangle / u' \;\; C.A.\;\; com \;\; \underline{E} \right\} \tag{35}$$

3 DETERMINATION OF MACROSCOPIC ELASTICITY FOR COMPOSITES WITH UNIDIRECTIONAL FIBERS

The unit cell is shown in Figure 3.

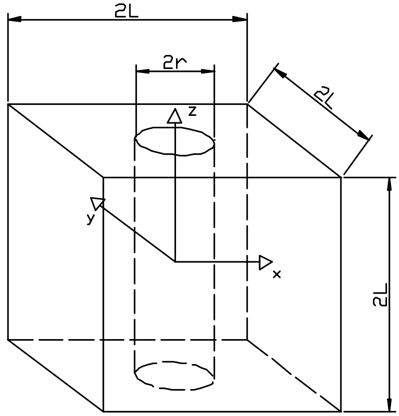

Figure 3: Composite unit cell with unidirectional fiber

The displacement field $\underline{u}^* = f(x, y)$ is approximated with periodic functions.

3.1 Approximation with Fourier series

Fourier series allows an approximation as good as needed. We write:

$$\begin{aligned} \underline{u}^* = & \sum_{i,j=0}^{i+j=p} A_{ij} \cos\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi y}{L}\right) + \sum_{i,j=0}^{i+j=p} B_{ij} \cos\left(\frac{i\pi x}{L}\right)\text{sen}\left(\frac{j\pi y}{L}\right) + \\ & + \sum_{i,j=0}^{i+j=p} C_{ij} \,\text{sen}\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi y}{L}\right) + \sum_{i,j=0}^{i+j=p} D_{ij} \,\text{sen}\left(\frac{i\pi x}{L}\right)\text{sen}\left(\frac{j\pi y}{L}\right) \end{aligned} \tag{36}$$

where

$$\underline{u}^*(L,y)=\underline{u}^*(-L,y) \quad \text{and} \quad \underline{u}^*(x,L)=\underline{u}^*(x,-L) \tag{37}$$

3.2 Approximation with (modified) Chebyshev polynomials

Polynomial ($T_n(x)$) of the first type will be used .They represent the solutions of the differential equation:

$$(1-x^2)y''-xy'+n^2y=0 \qquad n=0,\ 1,\ 2... \tag{38}$$

given by

$$T_n(x)=\cos(n\cos^{-1}x)=x^n-\binom{n}{2}x^{n-2}(1-x^2)+\binom{n}{4}x^{n-4}(1-x^2)^2-... \tag{39}$$

From (39):

$$\begin{array}{rl} T_0(x)=1 & T_1(x)=x \\ T_2(x)=2x^2-1 & T_3(x)=4x^3-3x \\ T_4(x)=8x^4-8x^2+1 & T_5(x)=16x^5-20x^3+5x \end{array} \tag{40}$$

and we observe that

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x) \tag{41}$$

Even Chebyshev polynomials satisfy naturally the periodicity criterion , as

$$T_{2k}(x)=T_{2k}(-x) \qquad k=0,\ 1,\ 2... \tag{42}$$

and thus $T'_{2k}(x)=T_{2k}(x)$, where $T'(x)$ is the set of polynomials to be used.

On the other hand, for the odd polynomials we have

$$T_{2k+1}(x)=-T_{2k+1}(-x) \tag{43}$$

and, particularly for the boundary of the unit cell

$$T_{2k+1}(1)=-T_{2k+1}(-1)=1 \tag{44}$$

However, using $T'_{2k+1}(x)=T_{2k+1}(x)-T_1(x)$ we have, from (44)

$$T'_{2k+1}(1)=T'_{2k+1}(-1)=0 \tag{45}$$

With this modification we have periodicity while maintaining orthogonality, because we are adding Chebyshev polynomials.

Then we write:

$$\underline{u}^* = \sum_{i,j=0}^{i+j=p} A_{ij} T'_i\left(\frac{x}{L}\right) T'_j\left(\frac{y}{L}\right) + \sum_{i,j=0}^{i+j=p} B_{ij} T'_i\left(\frac{y}{L}\right) T'_j\left(\frac{x}{L}\right) \tag{46}$$

4 HOMOGENIZATION IN VISCOELASTICITY

In this chapter, is presented a methodology to establish the homogenized constitutive matrix of a composite with time dependent mechanical properties. For non aging constituents, an incremental method is proposed.

4.1 Linear viscoelastic behavior law

The viscoelastic local constitutive law which relates stress and a prescribed history of strain is given by

$$\underline{\sigma}(t) = \int_{-\infty}^{+\infty} \underline{\underline{c}}(t,\tau) : \frac{\partial \underline{\varepsilon}(\tau)}{\partial \tau} d\tau = \underline{\underline{c}}(t,t) : \underline{\varepsilon}(t) - \int_{-\infty}^{t} : \frac{\partial \underline{\underline{c}}(t,\tau)}{\partial \tau} : \varepsilon(\tau) d\tau \tag{47}$$

The tensor $\underline{\underline{c}}$ (relaxation tensor) is assumed to be endowed with the same general symmetry properties as in elasticity:

$$c_{ijkl}(t,\tau) = c_{jikl}(t,\tau) = c_{ijlk}(t,\tau) = c_{klij}(t,\tau) \tag{48}$$

The inverse relation are defined as follows

$$\underline{\varepsilon}(t) = \int_{-\infty}^{+\infty} \underline{\underline{D}}(t,\tau) : \frac{\partial \underline{\sigma}(\tau)}{\partial \tau} d\tau = \underline{\underline{D}}(t,t) : \underline{\sigma}(t) - \int_{-\infty}^{t} \frac{\partial \underline{\underline{D}}(t,\tau)}{\partial \tau} : \sigma(\tau) d\tau \tag{49}$$

Symmetry properties are defined for the tensor $\underline{\underline{D}}$ (creep tensor):

$$D_{ijkl}(t,\tau) = D_{jikl}(t,\tau) = D_{ijlk}(t,\tau) = D_{klij}(t,\tau) \tag{50}$$

The argument t in $\underline{\underline{c}}$ and $\underline{\underline{D}}$ accounts for the instantaneous response while the argument τ characterizes the past histories of strain and stress respectively (the dependence on τ characterizes the delayed response).

The particular case of non aging materials corresponds to the time dependence form

$$\underline{\underline{c}}(t,\tau) = \underline{\underline{c}}(t-\tau) \text{ e } \underline{\underline{D}}(t,\tau) = \underline{\underline{D}}(t-\tau) \tag{51}$$

so that

$$\underline{\underline{c}}(t,t) = \underline{\underline{c}}(0) = \underline{\underline{c}}_0 \text{, } \underline{\underline{D}}(t,t) = \underline{\underline{D}}(0) = \underline{\underline{D}}_0 \text{ and } \underline{\underline{c}}_0 = \underline{\underline{D}}_0^{-1} \tag{52}$$

4.2 Localization problem

Let $\underline{E}(t)$ be the prescribed history of macroscopic strain field starting at *t=0*. The localization problem defined over the unit cell, whose solution gives the microscopic fields is:

$$\begin{gathered} div\,\underline{\sigma}(t) = 0\,,\ \underline{\sigma}(t).\underline{n} \text{ is antiperiodic} \\ \underline{u}(t) = \underline{E}(t).x + \underline{u}^*(t)\,,\ \underline{u}^*(t) \text{ is periodic} \\ \underline{\sigma}(t) = \int_{-\infty}^{+\infty} \underline{\underline{c}}(t,\tau) : \frac{\partial \underline{\varepsilon}(\tau)}{\partial \tau} d\tau \end{gathered} \tag{53}$$

$\underline{\varepsilon}$ is defined for $t \geq 0$. The initial conditions are fixed by setting all the fields to zero for $t<0$.

Once the localization problem (53) is solved, the macroscopic constitutive law relates $\underline{\Sigma}$ and $\underline{E}$. The linearity of the problem is described by the relation

$$\underline{\Sigma}(t) = \underline{\underline{C}}^{\hom}(t,t) : \underline{E}(t) - \int_{-\infty}^{t} \frac{\partial \underline{\underline{C}}^{\hom}(t,\tau)}{\partial \tau} : E(\tau) d\tau \tag{54}$$

where: $\underline{\underline{C}}^{\hom}(t,\tau)$ is the macroscopic relaxation tensor, and using the same principle:

$$\underline{E}(t) = \underline{\underline{D}}^{\hom}(t,t) : \underline{\Sigma}(t) - \int_{-\infty}^{t} \frac{\partial \underline{\underline{D}}^{\hom}(t,\tau)}{\partial \tau} : \Sigma(\tau) d\tau \tag{55}$$

where: $\underline{\underline{D}}^{\hom}(t,\tau)$ is the macroscopic creep tensor.

4.2.1 **Particular case: nonaging material**

As shown in (51), for the macroscopic relaxation tensor we have

$$\underline{\underline{C}}^{\hom}(t,\tau) = \underline{\underline{C}}^{\hom}(t-\tau) \tag{56}$$

Thus, only $\underline{\underline{C}}^{\hom}(t)$ must be determined and then is sufficient that

$$\underline{E}(t) = \underline{E} H(t) \tag{57}$$

so that (54) to becomes

$$\underline{\Sigma}(t) = \underline{\underline{C}}^{\hom}(t) : \underline{E} \tag{58}$$

4.3 Approximate solution (nonaging case)

In the constitutive relation (49), the strain can be divided in two parts:

$$\underline{\varepsilon}^{e}(t) = \underline{\underline{D}}_0 : \sigma(t) \text{, elastic part of } \underline{\varepsilon}(t) \tag{59}$$

$$\underline{\varepsilon}^{v}(t) = -\int_{\tau 0}^{t} \frac{\partial \underline{\underline{D}}(t,\tau)}{\partial \tau} : \underline{\sigma}(\tau) d\tau \text{, time dependent part of } \underline{\varepsilon}(t) \tag{60}$$

Therefore

$$\underline{\varepsilon}(t) = \underline{\varepsilon}^{e}(t) + \underline{\varepsilon}^{v}(t) \tag{61}$$

From the elastic part (59) is obtained (taking τ as integration variable)

$$\underline{\sigma}(\tau) = \underline{\underline{D}}_{0}^{-1} : \underline{\varepsilon}^{e}(\tau) = \underline{\underline{c}}_{0} : \left(\underline{\varepsilon}(\tau) - \underline{\varepsilon}^{v}(\tau)\right) \tag{62}$$

The localization problem is solved subdividing the time interval [0;T] in increments Δt. Assuming that $\underline{\sigma}(\tau)$ and $\underline{\varepsilon}(\tau)$ are known for any $\tau \in [0;t]$, in the interval $\tau \in]t;t+\Delta t]$, (61) can be approximated, for Δt sufficiently short:

$$\underline{\sigma}(\tau) \cong \underline{\underline{c}}_{0} : \left(\underline{\varepsilon}(\tau) - \underline{\varepsilon}^{v}(t)\right) \tag{63}$$

where $\underline{\varepsilon}^{v}(t)$ is predetermined.

Consequently, over an interval $]t;t+\Delta t]$, formally there is an elasticity problem with initial given strain. For $\forall \tau \in]t;t+\Delta t]$:

$$\begin{gathered} div\underline{\sigma}(\tau) = 0 \text{ , } \underline{\sigma}(\tau).\underline{n} \text{ is antiperiodic} \\ \underline{u}(\tau) = \underline{E}(\tau).x + \underline{u}^{*}(\tau) \text{ , } \underline{u}^{*}(\tau) \text{ is periodic} \\ \underline{\sigma}(\tau) = \underline{\underline{c}}_{0} : \left(\underline{\varepsilon}(\tau) - \underline{\varepsilon}^{v}(t)\right) \end{gathered} \tag{64}$$

Remarkably:

$$\langle \underline{\varepsilon}(\tau) \rangle = \underline{E}(\tau) = \underline{E}_{0} \tag{65}$$

Since $\underline{E}$ and $\underline{\varepsilon}^{v}$ are not time dependent over the considered interval, $\underline{u}(\tau)$, $\underline{\sigma}(\tau)$ and $\underline{\varepsilon}(\tau)$ are independent of τ. Then, the constitutive relation (63) is written:

$$\underline{\sigma} = \underline{\underline{c}}_{0} : \left(\underline{\varepsilon} - \underline{\varepsilon}^{v}(t)\right) \tag{66}$$

According the previous Section the solution $\underline{\varepsilon}$ from (64) is determined minimizating a functional $\langle \Psi \rangle$. The elastic potential Ψ can be defined in such way that:

$$\frac{\partial \Psi(\underline{\varepsilon})}{\partial \underline{\varepsilon}} = \underline{\sigma} = \underline{\underline{c}}_{0} : (\underline{\varepsilon} - \underline{\varepsilon}^{v}) \tag{67}$$

Integrating with relation to $\underline{\varepsilon}$:

$$\Psi(\underline{\varepsilon}) = \frac{1}{2}(\underline{\varepsilon} - \underline{\varepsilon}^{v}) : \underline{\underline{c}}_{0} : (\underline{\varepsilon} - \underline{\varepsilon}^{v}) = \frac{1}{2}\underline{\varepsilon} : \underline{\underline{c}}_{0} : \underline{\varepsilon} - \underline{\varepsilon} : \underline{\underline{c}}_{0} : \underline{\varepsilon}^{v} + \frac{1}{2}\underline{\varepsilon}^{v} : \underline{\underline{c}}_{0} : \underline{\varepsilon}^{v} \tag{68}$$

The last term of (68) is unnecessary, since it is constant. As minimization solution are obtained:

$$\underline{\varepsilon}(t + \Delta t) = \underline{\varepsilon} \tag{69}$$

$$\underline{\sigma}(t+\Delta t)=\underline{\sigma} \text{ (from (66))} \tag{70}$$

To calculate $\underline{\varepsilon}^{v}(t+\Delta t)$ state variables are used, as shown in the next Section [4.3.1].

The process is repeated over the interval $]t+\Delta t; t+2\Delta t]$ taking $\underline{\varepsilon}^{v}(\tau)\cong\underline{\varepsilon}^{v}(t+\Delta t)$ determined for the end of the previous interval.

The matrix $\underline{\underline{C}}^{\text{hom}}(t+\Delta t)$ is obtained from the relation

$$\langle\sigma\rangle=\underline{\underline{C}}^{\text{hom}}:\underline{E} \tag{71}$$

4.3.1 Representation of $\underline{\varepsilon}^{v}$ by means of state variables

Expression (17) be written for non aging case as

$$\underline{\varepsilon}^{v}(t)=-\int_{\tau 0}^{t}\frac{\partial\underline{\underline{D}}(t-\tau)}{\partial\tau}:\underline{\sigma}(\tau)d\tau \tag{72}$$

Let us consider first a one dimensional case and approximate the function $-\dfrac{\partial D(t-\tau)}{\partial\tau}$ by means of the exponential series (often called Dirichlet-Prony series):

$$-\frac{\partial D(t-\tau)}{\partial\tau}\cong\sum_{i=1}^{n}\frac{1}{\eta_i}e^{\frac{-(t-\tau)}{\theta_i}} \tag{73}$$

where: $\theta_i=\dfrac{\eta_i}{E_i}$ corresponding to the generalized Kelvin model (elements with elasticity E_i and viscosity η_i). Such an approximation can be as good as we like, depending on the number of terms included. We introduce then the n values

$$\varepsilon^{v}(t)\cong\sum_{i=1}^{n}q_i(t)=\sum_{i=1}^{n}\int_{\tau 0}^{t}\frac{1}{\eta_i}e^{\frac{-(t-\tau)}{\theta_i}}\sigma(\tau)d\tau \tag{74}$$

The state variable q_i has the following integral expression:

$$q_i(t)=\int_{\tau 0}^{t}\frac{1}{\eta_i}e^{\frac{-(t-\tau)}{\theta_i}}\sigma(\tau)d\tau \tag{75}$$

Integrating within time interval [0; $t+\Delta t$], (75) may be written in the form:

$$q_i(t+\Delta t)=\int_{0}^{t}\frac{1}{\eta_i}e^{\frac{-(t+\Delta t-\tau)}{\theta_i}}\sigma(\tau)d\tau+\int_{t}^{t+\Delta t}\frac{1}{\eta_i}e^{\frac{-(t+\Delta t-\tau)}{\theta_i}}\sigma(\tau)d\tau \tag{76}$$

The first integral is equal to $e^{\frac{-\Delta t}{\theta_i}} q_i(t)$; the second one becomes, assuming $\frac{\sigma(\tau)}{\eta_i}$ constant over the interval:

$$\frac{\sigma(t+\Delta t)}{\eta_i} e^{\frac{-(t+\Delta t)}{\theta_i}} \int_t^{t+\Delta t} \frac{1}{\eta_i} e^{\frac{\tau}{\theta_i}} \sigma(\tau) d\tau = \frac{\sigma(t+\Delta t)}{\eta_i} \theta_i \left(1 - e^{\frac{-\Delta t}{\theta_i}}\right) \tag{77}$$

Thus for the equation (76), with a sufficiently short Δt , we have

$$q_i(t+\Delta t) = e^{\frac{-\Delta t}{\theta_i}} q_i(t) + \frac{\sigma(t+\Delta t)}{E_i} \left(1 - e^{\frac{-\Delta t}{\theta_i}}\right) \tag{78}$$

In the 3-D case (orthotropic materials), we associate different state variables to each element of $\underline{\sigma}$ to determine $\underline{\varepsilon}^v$.

4.4 Description of the iterative process

The problem is determinate $\underline{\underline{C}}^{\hom}(t)$ for $t \in [0;T]$. We may divide the time interval $[0;T]$ in such way that

$$\Delta t = t_{n+1} - t_n \tag{79}$$

To begin the process, at $t=0$, first we calculate the instantaneous elasticity $\underline{\underline{C}}_0^{\hom}$ and then we estimate $\underline{\varepsilon}_1^v$, taking $\underline{\varepsilon}^v = 0$.

0) $t=0$ $\quad \underline{\varepsilon}_0^v = 0 \rightarrow \min\langle\Psi\rangle \rightarrow \underline{\varepsilon}_1 \rightarrow \underline{\sigma}_1 = \underline{\underline{c}}_0 : \underline{\varepsilon}_1 \rightarrow \underline{\varepsilon}_1^v \rightarrow \underline{\underline{C}}_0^{\hom}$

1) $]0;t_1]$ $\quad \underline{\varepsilon}_1^v \rightarrow \min\langle\Psi\rangle \rightarrow \underline{\varepsilon}_2 \rightarrow \underline{\sigma}_2 = \underline{\underline{c}}_0 : \left(\underline{\varepsilon}_2 - \underline{\varepsilon}_1^v\right) \rightarrow \underline{\varepsilon}_2^v \rightarrow \underline{\underline{C}}^{\hom}(t_1)$

n+1) $]t_n;t_{n+1}]$ $\quad \underline{\varepsilon}_n^v \rightarrow \min\langle\Psi\rangle \rightarrow \underline{\varepsilon}_{n+1} \rightarrow \underline{\sigma}_{n+1} = \underline{\underline{c}}_0 : \left(\underline{\varepsilon}_{n+1} - \underline{\varepsilon}_n^v\right) \rightarrow \underline{\varepsilon}_{n+1}^v \rightarrow \underline{\underline{C}}^{\hom}(t_{n+1})$

When $t_{n+1}=T$ the iterative process is completed.

5 EXAMPLES IN ELASTICITY

Now we present some elasticity results for a one-dimension composite.

Component properties:

Matrix: $E_m = 3{,}12 \ GPa$ $\quad \nu_m = 0{,}38$

Reinforcement: $E_r = 72{,}4 \ GPa$ $\quad \nu_r = 0{,}22$

n is the volumetric relation between the reinforcement and the unit cell

5.1 Comparison Data

PMF – Periodic Microstructure Formulation (E. J. Barbero)
Halpin-Tsai – only E_x
EF – Finite Element – only E_x and G_{xy}

(The Finite Element model is composed by 200 unit cells in the direction transversal to the fibers; plane stress – analysis with software Ansys.
For E_x – 10x20 cells – unit traction - vertical direction
For G_{xy} – 10x20 cells – unit shear - largest dimension
Due the reduced number of cells to represent the composite periodicity these are approximate models.

5.2 Procedures and Results

First, the relative efficiency of Fourier Chebyshev expansions is analyzed comparing expansions with the same number of terms.

Fourier:	p = 1	Chebyshev:	p = 3	(2 terms)
	p = 2		p = 5	(6 terms)
	p = 3		p = 7	(12 terms)
	p = 4		p = 9	(20 terms)

The following graphics are normalized with relation to the matrix properties.

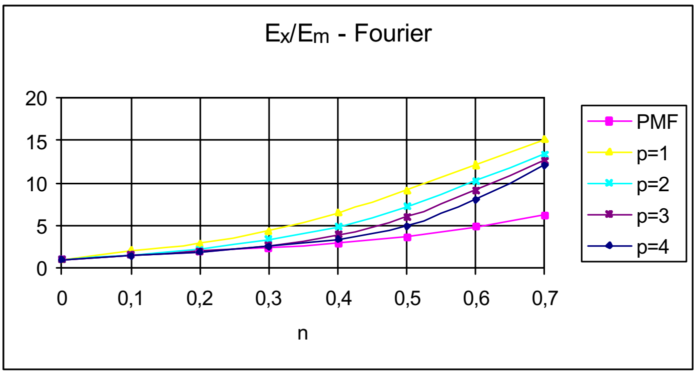

Figure 4: Transversal Elasticity Modulus (Fourier)

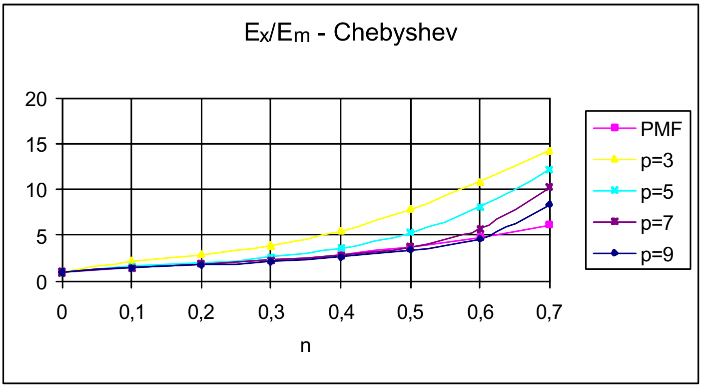

Figure 5: Transversal Elasticity Modulus (Chebyshev)

We notice that for transverse elastic modulus Chebyshev polynomials converge faster than Fourier series to the PMF results. Shear moduli show similar behavior. The longitudinal elastic modulus is linear with respect to n and it is independent to the expansion.

Chebyshev polynomials have computational advantages; the integration process is easier and faster.

Now we present the calculated results with p=13 (polynomials up to 13^{th} degree or less). It is a good approximation for $n \leq 0{,}6$. Polynomials with higher degree are needed for higher values of n.

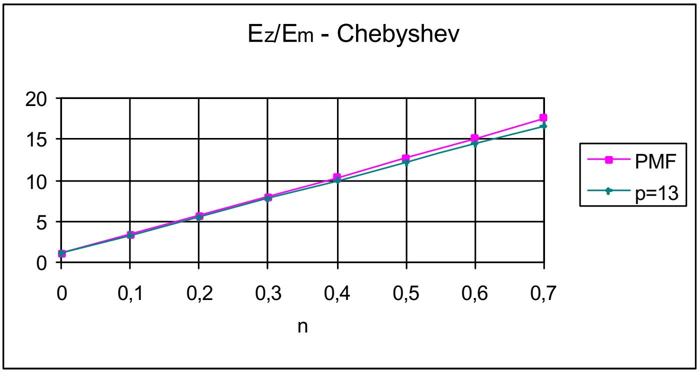

Figure 6: Longitudinal Elasticity Modulus

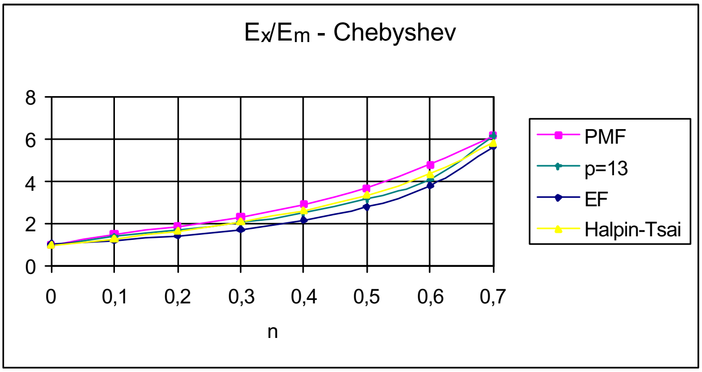

Figure 7: Transversal Elasticity Modulus

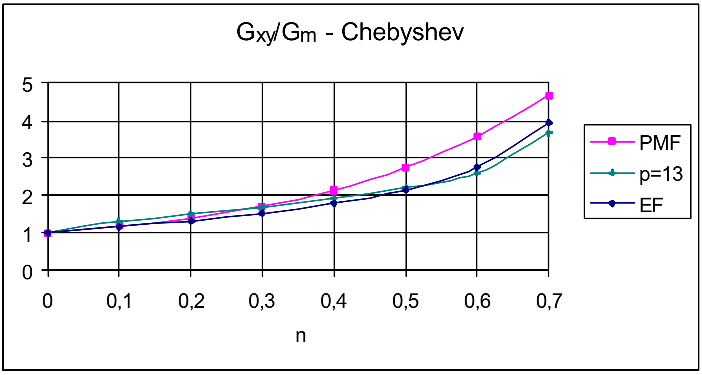

Figure 8: Transversal Shear Modulus

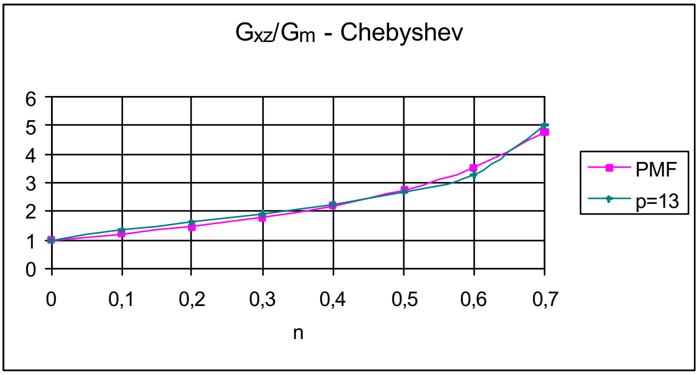

Figure 9: Longitudinal Shear Modulus

6 EXAMPLES IN VISCOELASTICITY

Now we present an academic example. The fiber is considered elastic: $E_r = 72{,}4 \;\; GPa$. The matrix is modeled as a Maxwell material: $E_m = 3{,}12 \;\; GPa$; $\eta_m = 31{,}12 \;\; GPa.day$. With the Poisson coefficient taken as zero, the behavior should correspond to a Zener model, which is used as comparison.

Figure 10 shows the evolution of the convergence (relaxation function) with the size of Δt , without fiber (Maxwell model).

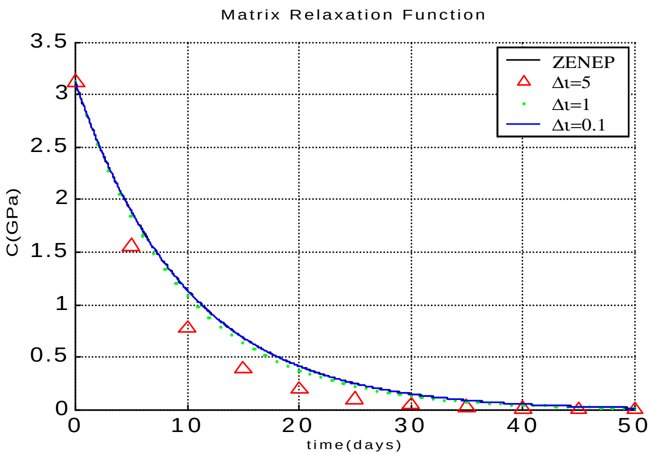

Figure 10: Convergence evolution with Δt (days)

Figure 11 shows the inclusion of fiber aspect. It is used $\Delta t = 0.1 day$; n is the volumetric relation between the fiber and the unit cell

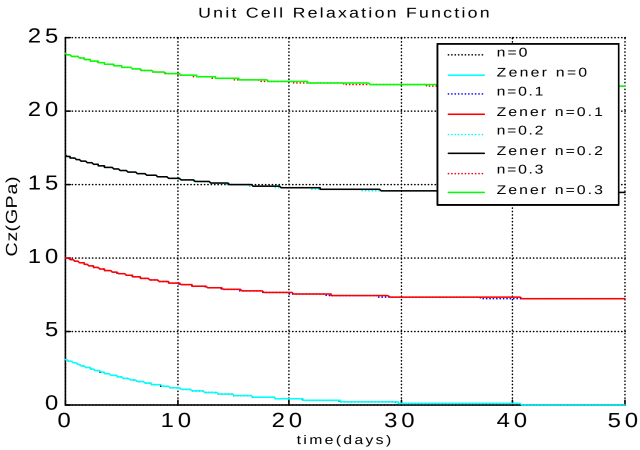

Figure 11: Inclusion of fiber

7 CONCLUSIONS

A procedure for the determination of homogenized properties of fiber composites has been presented. The use of Fourier series follows the work of Maghous[8]. The use of Chevishev polynomials, that seems to be new for this application, gives the same results with more efficiency. Results are validated by means of comparison with other references. The procedure seems to be an efficient alternative to Finite Element procedures, particularly in the viscoelastic case.

8 ACKNOWLEDGEMENT

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